Theory of Equation 5B

2. calculate Sturm's functions and locate the position of the real roots of the equ^n

(i) $x^3 - 3x - 1 = 0$

A:- Given equ^n is $x^3 - 3x - 1 = 0$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$.

$f(x) = x^3 - 3x - 1$

$f_1(x) = x^2 - 1$

$f_2(x) = 2x + 1$

$f_3(x) = 3$

$x^2-1 \,)\, x^3 - 3x - 1 \,(\, x$
$x^3 - x$
$-2x - 1$
$-\;2x+1$

$2x+1 \,)\, x^2 - 1 \,(\, x - 1$
$2x^2 - 2$
$2x^2 + x$
$-x - 2$
$-2x - 4$
$-2x - 1$
-3
3

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of sign
$-\infty$	$-$	$+$	$-$	$+$	3
0	$-$	$-$	$+$	$+$	1
∞	$+$	$+$	$+$	$+$	0

The equ^n has three real roots, two negative and one +ve.

Location of roots

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of sign
		$+$	$-$	$+$	3
-2	$-$	0	$-$	$+$	2
-1	$+$	$-$	$+$	$+$	1
0	$-$	0	$+$	$+$	1
1	$-$	$+$	$+$	$+$	0
2	$+$				

one +ve root lies between $(1,2)$ and two -ve roots lie $(-1,0)$, $(-2,-1)$.

(ii) $x^3 - 7x + 7 = 0$

A:- Given equ^n is $x^3 - 7x + 7 = 0$

$f'(x) = 3x^2 - 7$

$f(x) = x^3 - 7x + 7$

$f_1(x) = 3x^2 - 7$

$f_2(x) = 2x - 3$

$f_3(x) = 1$

$3x^2 - 7 \,)\, x^3 - 7x + 7 \,(\, x$
3
$3x^3 - 21x + 21$
$3x^3 - 7x$
$-7 \;|\; -14x + 21$
$2x - 3$

$2x-3 \,)\, 3x^2 - 7 \,(\, 3x + 9$
2
$6x^2 - 14$
$6x^2 - 9x$
$9x - 14$
2
$18x - 28$
$18x - 27$
-1
1

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of signs
$-\infty$	$-$	$+$	$-$	$+$	3
0	$+$	$-$	$-$	$+$	2
∞	$+$	$+$	$+$	$+$	0

The equ^n has one negative and two +ve roots.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of signs
-2					
-1	$+$	$-$	$-$	$+$	2
0	$+$	$-$	$-$	$+$	2
1	$+$	$-$	$-$	$+$	2
2	$+$	$+$	$+$	$+$	0
$\frac{3}{2}$	$-$	$+$	0	$+$	1

$\therefore$ Two +ve roots lies between $(1, \frac{3}{2}), (\frac{3}{2}, 2)$ and one negative root lies between.

3. Show that the equ^n^s have no real root.

(i) $x^4 - x + 3 = 0$.

A:- The given equ^n is $f(x) = x^4 - x + 3$ $\quad f'(x) = 4x^3 - 1$

$f(x) = x^4 - x + 3$

$f_1(x) = 4x^3 - 1$

$f_2(x) = x - 4$

$f_3(x) = \text{constant} = -255$

$$4x^3 - 1 \,)\, x^4 - x + 3 \,(\, x$$
$$\times 4: \quad 4x^4 - 4x + 12$$
$$4x^4 - x$$
$$-3 \,(\, -3x + 12$$
$$x - 4$$

$$x - 4 \,)\, 4x^3 - 1 \,(\, 4x^2 + 16x + 64$$
$$4x^3 - 16x^2$$
$$16x^2 - 1$$
$$16x^2 - 64x$$
$$64x - 1$$
$$64x - 256$$
$$-\,(\,-255$$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of sign
$-\infty$	+	−	−	−	1
0	+	−	−	−	1
∞	+	+	+	−	1

∴ The equ^n $x^4 - x + 3 = 0$ have no real root. A

(ii) $x^6 - x + 6 = 0$.

A:- The given equ^n is $x^6 - x + 6 = 0$

$f(x) = x^6 - x + 6$

$f_1(x) = 6x^5 - 1$

$f_2(x) = 5x - 36$

$f_3(x) = \text{constant}$

$$6x^5 - 1 \,)\, x^6 - x + 6 \,(\, x$$
$$\times 6: \quad 6x^6 - 6x + 36$$
$$6x^6 - x$$
$$-\,(\,-5x + 36$$
$$5x - 36$$

$$5x - 36 \,)\, 6x^5 - 1 \,(\, 6x^4 + 216x^3$$
$$\times 5: \quad 30x^5 - 5$$
$$30x^5 - 216x^4$$
$$216x^4 - 5$$
$$\times 5: \quad 1080x^4 - 25$$
$$1080x^4 - 7776x^3$$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of sign
$-\infty$	+	−	−	−	1
0	+	−	−	−	1
∞	+	+	+	−	1

The equ^n $x^6 - x + 6 = 0$ have no real root. A.

(iii) $x^4 + 3x^3 - x^2 - 3x + 11 = 0$

A:- The given equ^n is $x^4 + 3x^3 - x^2 - 3x + 11 = 0$

Let $f(x) = x^4 + 3x^3 - x^2 - 3x + 11$

$f_1(x) = 4x^3 + 9x^2 - 2x - 3$

$f_2(x) = 7x^2 + 6x - 37$

$f_3(x) = -44x - 81$

$f_4(x) = \text{constant} > 0$

$$4x^3 + 9x^2 - 2x - 3 \,)\, x^4 + 3x^3 - x^2 - 3x + 11 \,(\, x + 3$$
$$\times 4: \quad 4x^4 + 12x^3 - 4x^2 - 12x + 44$$
$$4x^4 + 9x^3 - 2x^2 - 3x$$
$$3x^3 - 2x^2 - 9x + 44$$
$$\times 4: \quad 12x^3 - 8x^2 - 36x + 176$$
$$12x^3 + 27x^2 - 6x - 9$$
$$-\,(\,-35x^2 - 30x + 167$$
$$35x^2 + 30x - 167$$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	Change of sign
$-\infty$	+	−	+	+	+	2
0	+	−	−	−	+	2
∞	+	+	+	−	+	2

∴ The equ^n have no real root. A

4. If a and b are positive prove that the equ^n $x^5 - 5ax + 4b = 0$ has three real roots or only one according as $a^5 >$ or $< b^4$.

A:- $f(x) = x^5 - 5ax + 4b$ $\quad f'(x) = 5x^4 - 5a$

$f_1(x) = x^4 - a$ $\quad = 5(x^4 - a)$

$f_2(x) = ax - b$

$f_3(x) =$ constant

$f_2(x) = 0$ gives $x = \frac{b}{a}$

$$x^4 - a \,\big)\, x^5 - 5ax + 4b \,\big(\, x$$
$$\underline{x^5 \quad - ax}$$
$$-4 \big(-4ax + 4b$$
$$ax - b$$

$\therefore f_1\left(\frac{b}{a}\right)$ and $f_3\left(\frac{b}{a}\right)$ are of opposite signs.

$\therefore f_1\left(\frac{b}{a}\right) = \left(\frac{b}{a}\right)^4 - a = \frac{b^4 - a^5}{a^4}$

<u>Case :- 1</u> $f_1\left(\frac{b}{a}\right) > 0$ i.e $\frac{b^5 - a^5}{a^4} > 0$ or.

$b^4 > a^5$

$\therefore f_3\left(\frac{b}{a}\right) < 0 \quad \therefore f_3(x) < 0 \quad \forall x \in \mathbb{R}$.

Now

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of signs
$-\infty$	$-$	$+$	$-$	$-$	2
0	$+$	$-$	$-$	$-$	1
∞	$+$	$+$	$+$	$-$	1

$\therefore f(x) = 0$ has only one real negative root.

<u>Case 2:-</u> $f_1\left(\frac{b}{a}\right) < 0$ i.e $\frac{b^4 - a^5}{a^4} < 0$ or.

$b^4 < a^5$

$\therefore f_3\left(\frac{b}{a}\right) > 0 \quad \therefore f_3(x) > 0 \quad \forall x \in \mathbb{R}$

Now,

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	change of signs
$-\infty$	$-$	$+$	$-$	$+$	3
0	$+$	$-$	$-$	$+$	2
∞	$+$	$+$	$+$	$+$	0

The eqn has 3 real roots, two +ve and one negative.

$\therefore$ The given eqn has three real roots if $a^5 > b^4$ and one real root if $a^5 < b^4$.

5. Apply Descartes' rule of signs to find the nature of the roots of the eqn (i) $x^4 + 2x^2 + 3x - 1 = 0$.

A:- Given eqn is $x^4 + 2x^2 + 3x - 1 = 0$

Let $f(x) = x^4 + 2x^2 + 3x - 1$

The sign in the sequence of co-efficient of $f(x)$ are

$+ \quad + \quad + \quad -$

There are only one variation of sign and therefore the no. of +ve roots of $f(x) = 0$ is exactly one.

Now, $f(-x) = x^4 + 2x^2 - 3x - 1$

The signs in the sequence of co-efficient of $f(-x)$ are

$+ \quad + \quad - \quad -$

There are only one variation of sign and therefore the no of -ve roots of $f(-x)=0$ is exactly one.

; the eqⁿ has no zero root. Therefore the no of real root is $1+1=2$. the eqⁿ being of degree 4 has four roots.

The no of complex roots $f(x)=0$ is $4-2=2$, A

(ii) $x^8+1=0$

A :- The given eqⁿ is $x^8+1=0$.

Let $f(x)=x^8+1$.

The sign in the sequence of co-efficient of $f(x)$ are + +.

There is no variation of sign and therefore the number of +ve real root is zero.

$f(-x)=x^8+1$

The sign in the sequence of co-efficient of $f(x)$ are + +.

There is no negative real root. The eqⁿ has no zero root.

∴ The eqⁿ has 8 complex roots.

(iii) $x^{10}-1=0$

A :- The given eqⁿ is $x^{10}-1=0$

Let $f(x)=x^{10}-1$.

The signs in the sequence of co-efficient of $f(x)$ are + -.

There is only one variation of sign and therefore the no of +ve root of $f(x)=0$ is exactly one.

$f(-x)=x^{10}-1$.

The signs in the sequence of the co-efficient of $f(-x)$ are + -.

There is only one variation of sign and therefore the no of -ve root $f(-x)=0$ is exactly one.

The eqⁿ has no zero roots,

∴ The no of real roots is 2 and no of complex roots are 8. A

(iv) $x^7+x^5-x^3=0$

A :- The given eqⁿ is $x^7+x^5-x^3=0$.

Let $f(x)=x^7+x^5-x^3$

The signs in the sequence of the co-efficients of $f(x)$ are + + -.

The is only one variation of sign and therefore the no of +ve root is exactly one.

$f(-x)=-x^7-x^5+x^3$

The sign in the sequence of the co-efficients of $f(-x)$ are - - +

There is only one variation of signs. Therefore the no of -ve root is exactly one.

∴ The no of real roots are 2.

The equ^n can be written as $x^3(x^4 + x^2 - 1) = 0$.

∴ The equ^n has 3 zero roots.

∴ The no of real roots are 5 and complex roots are 2. A

6. Apply Descartes' rule of signs to ascertain the minimum no of (i) complex roots of the equ^n (i) $x^6 - 3x^2 - 2x - 3 = 0$.

A:- Given equ^n is $x^6 - 3x^2 - 2x - 3 = 0$

Let $f(x) = x^6 - 3x^2 - 2x - 3$.

The sign in the sequence of $f(x)$ are + - - -.

There is only one variation of sign and therefore the no of +ve root is exactly one.

$f(-x) = x^6 - 3x^2 + 2x - 3$.

The sign in the sequence of $f(-x)$ are + ~~+~~ - + -.

There are three variation of sign and therefore the no of -ve root atmost 3.

$f(x) = 0$ has no zero roots.

no of +ve roots	no of -ve roots	no of zero roots	degree of equ^n	no of complex roots
1	3	0	6	2
1	1	0	6	4

∴ Minimum no of complex roots are 2. A

(ii) $x^7 - 3x^3 - x + 1 = 0$.

A:- Given equ^n is $x^7 - 3x^3 - x + 1 = 0$

Let $f(x) = x^7 - 3x^3 - x + 1$.

The signs in the sequence of $f(x)$ are + - - +.

There is only two variation of sign and therefore the no of +ve root may have two / atmost two.

$f(-x) = -x^7 + 3x^3 + x + 1$.

The signs in the sequence of $f(-x)$ are - + + +.

There are only one variation of sign and therefore the no of

-ve root is exactly one.

no of +ve roots	no of -ve roots	no of zero roots	degree of eq^n	no of complex
2	1	0	7	4
0	1	0	7	6

$\therefore$ Minimum no of complex roots are 4. A

(iii) $x^7 - 3x^3 + x^2 = 0$

A:- Let $f(x) = x^7 - 3x^3 + x^2$.

The signs in the sequence of $f(x)$ are $+ \; - \; +$.
There are two variation of sign and therefore no of +ve roots may have two.

$f(-x) = -x^7 + 3x^3 + x^2$.

The signs in the sequence of $f(-x)$ are $- \; + \; +$,
There are only one variation of sign and therefore the no of -ve root is exactly one.

no of +ve roots	no of -ve roots	no of zero roots	degree of eq^n	no of complex roots
2	1	2	7	2
0	1	2	7	4

$\therefore$ Minimum no of complex roots are 2. A

Ex-5C

1. Solve the eq^{ns} (i) $x^3 + 6x^2 - 3x - 18 = 0$ given that the sum of two of the roots is zero.

A:- Given eq^n is $x^3 + 6x^2 - 3x - 18 = 0$ (1)
Let α, β, γ are the roots of the eq^n (1) by the given con^n.

$\alpha + \beta = 0$ (2)

From the relation between roots and co-efficients we get,

$\alpha + \beta + \gamma = -6$ (3)

$\alpha\beta + \beta\gamma + \gamma\alpha = 3$ (4)

$\alpha\beta\gamma = 18$ (5)

From (2) we get, $\gamma = -6$

From (5) " " $\alpha\beta = -3$

From (2) we get,

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$
$= (0)^2 - 4.(-3)$
$= 12$

$\alpha - \beta = 2\sqrt{3}$
$\alpha + \beta = 0$
$2\alpha = 2\sqrt{3}$
or, $\alpha = \sqrt{3}$ $\quad \beta = -\sqrt{3}$

$\therefore$ The roots of the eq^n are
$\sqrt{3}, -\sqrt{3}, -6$ A

(ii) $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$

A:- The given eq^n is $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ (1)
Let $\alpha, \beta, \gamma, \delta$ are the roots of the eq^n (1).
Then con^n is $\alpha + \beta = 0$ (2)

Exercise - 5C

From the relation between roots and co-efficients,

$\alpha+\beta+\gamma+\delta = 2$ ③

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta = 4$ ④

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = -6$ ⑤

$\alpha\beta\gamma\delta = -21$ ⑥

From ④ we get

$\alpha\beta+\gamma\delta = 4$

From ③ we get,

$\gamma+\delta = 2$ ⑦

From ⑥ we get,

$\gamma\delta = \frac{-21}{-3} = 7$

From ⑤ we get,

$2\alpha\beta + 0 = -6$

or, $\alpha\beta = -3$

Now, $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n,

$[t^2-(\alpha+\beta)t+\alpha\beta][t^2-(\gamma+\delta)t+\gamma\delta] = 0$

or, $(t^2-3)(t^2-2t+7) = 0$

or, $t^2 = 3$

$t = \pm\sqrt{3}$

$t = \frac{2\pm\sqrt{4-4.1.7}}{2.1}$

$= 1\pm\sqrt{6}i$

∴ The roots of the equ^n are $\pm\sqrt{3}$, $1\pm\sqrt{6}i$

(iii) $2x^4+8x^3+3x^2+6x+1=0$

A:- Given equ^n is $2x^4+8x^3+3x^2+6x+1=0$ ①

Let α, β, γ are the roots of the equ^n ① by the given con^n $\alpha+\beta=0$

From the relation between roots and co-efficients we get,

$\alpha+\beta+\gamma+\delta = 8/2$ ③

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta = \frac{3}{2}$ ④

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = -\frac{6}{2}$ ⑤

$\alpha\beta\gamma\delta = \frac{1}{2}$ ⑥

From ③

$\gamma+\delta = -4$ ⑦

From ⑤,

$-4\alpha\beta = -\frac{4}{2}$

or, $\alpha\beta = \frac{1}{2}$

From ⑥, $\gamma\delta = \frac{1}{2}\times\frac{2}{1} = 1$,

Now, $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n,

$[t^2-(\alpha+\beta)t+\alpha\beta][t^2-(\gamma+\delta)t+\gamma\delta] = 0$

or, $(t^2+\frac{1}{2}) = 0$

$t = \pm\frac{1}{\sqrt{2}}i$

$t^2+4t+1 = 0$

$t = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3}$

∴ The roots of the equ^n are $\pm\frac{i}{\sqrt{2}}$, $-2\pm\sqrt{3}$

2. Solve the equ^ns

(i) $x^3+5x^2+7x+2=0$ given that the product of two of the roots is 1

A:- Given equ^n is $x^3+5x^2+7x+2=0$ ①

Let α, β, γ are the roots of the equ^n ① by the given con^n $\alpha\beta=1$ ②

From the relation between roots and co-efficients we get,

$\alpha+\beta+\gamma = -5$ ②

$\alpha\beta+\beta\gamma+\gamma\alpha = 7$ ③

$\alpha\beta\gamma = -2$ ④

$\gamma = -2$

$\alpha\beta + (\alpha+\beta)\gamma = 7$

or, $1 + \gamma(\alpha+\beta) = 7$

or, $\gamma(\alpha+\beta) = -6$

or, $\alpha+\beta = -3$ (6)

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

or, $\alpha - \beta = \sqrt{5}$

$\alpha - \beta = \sqrt{5}$

$\alpha + \beta = -3$

$2\alpha = \sqrt{5} - 3$

or, $\alpha = \dfrac{\sqrt{5}-3}{2}$, $\beta = \dfrac{-3-\sqrt{5}}{2}$

The roots of the equ^n are -2, $\dfrac{-3 \pm \sqrt{5}}{2}$, Ans.

(ii) $x^4 + 2x^3 + 5x^2 + 4x + 3 = 0$

A:- Given equ^n is $x^4 + 2x^3 + 5x^2 + 4x + 3 = 0$ (1)

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n (1) by the given con^n $\alpha\beta = 1$ (2)

From the relation between roots and coefficients we get,

$\alpha+\beta+\gamma+\delta = -2$ (3)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = 5$ (4)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -4$ (5)

$\alpha\beta\gamma\delta = -3$ (6)

From (2) we get,

$\gamma\delta = -3$ (7)

From (5) we get,

$1(\gamma+\delta) + 3(\alpha+\beta) = -4$

or, $\gamma+\delta = -4 - 3(\alpha+\beta)$ (8)

$(\alpha+\beta) + (\gamma+\delta) = -2$

or, $\alpha+\beta + (-3\alpha - 3\beta) - 4 = -2$

or, $\alpha+\beta = -1$

From (8) we get

$\gamma + \delta = -1$

Now, $\alpha, \beta, \delta, \gamma$ are the roots of the equ^n,

$[t^2 - (\alpha+\beta)t + \alpha\beta]$ $[t^2 - (\gamma+\delta)t + \gamma\delta] = 0$

or, $(t^2 + t + 1)$ $(t^2 + t + 3) = 0$

or, $t^2 + t + 1 = 0$

$t = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$= \dfrac{-1 \pm i\sqrt{3}}{2}$

$t^2 + t + 3 = 0$

$t = \dfrac{-1 \pm \sqrt{1-12}}{2} = \dfrac{-1 \pm \sqrt{11}\,i}{2}$

$\therefore$ Required roots of the equ^n are

$\dfrac{-1 \pm i\sqrt{3}}{2}$ and $\dfrac{-1 \pm i\sqrt{11}}{2}$. Ans

(iii) $2x^4 + 2x^3 - 33x^2 - 10x + 5 = 0$

A:- Given equ^n is $2x^4 + 2x^3 - 33x^2 - 10x + 5 = 0$ (1)

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n (1)

By the given con^n, $\alpha\beta = 1$ (2)

From the relation between roots and co-efficients we get,

$\alpha+\beta+\gamma+\delta = -1$ (3)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = -33$ (4)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = \dfrac{10}{2}$ (5)

$\alpha\beta\gamma\delta = \dfrac{5}{2}$ (6)

From (6)

$\gamma\delta = \frac{5}{2}$ (7)

From (5) we get,

$1(\gamma+\delta) + \frac{5}{2}(\alpha+\beta) = 5$

or, $(\gamma+\delta) + \frac{5}{2}(-\delta-\gamma-1) = 5$

or, $\delta+\gamma = -5$ (8)

$\therefore \alpha+\beta = -1-\gamma-\delta$

From (8) we get

$\therefore \alpha+\beta = -1(+5) = 4$

Now, $\alpha\beta, \gamma\delta$ are the roots of the equ^n,

$[t^2-(\alpha+\beta)t+\alpha\beta][t^2-(\gamma+\delta)t+\gamma\delta] = 0$

or $[t^2-4t+1][t^2+5t+\frac{5}{2}] = 0$

$t^2-4t+1 = 0$

$t = \frac{4 \pm \sqrt{16-4}}{2}$

$= 2 \pm \sqrt{3}$

$t^2+5t+5/2 = 0$

$t = \frac{-5 \pm \sqrt{25-4.1.5/2}}{2.1}$

$= \frac{-5 \pm \sqrt{15}}{2}$

$\therefore$ The roots of the equ^n are $2 \pm \sqrt{3}$ and $\frac{-5 \pm \sqrt{15}}{2}$ A

3. (i) Solve the equ^{n}s given that the roots are in arithmetic progression. $x^3+6x^2+11x+6 = 0$

A:– The given equ^n is $x^3+6x^2+11x+6 = 0$ (1)

Let the roots are $\alpha-\beta, \alpha, \alpha+\beta$,

From the relation between roots and co-efficients

$\alpha-\beta+\alpha+\alpha+\beta = -6$ (2)

$(\alpha-\beta)\alpha + \alpha(\alpha+\beta) + (\alpha+\beta)(\alpha-\beta) = 11$ (3)

$\alpha(\alpha-\beta)(\alpha+\beta) = -6$ (4)

From (2), $\alpha = -2$

From (3) we get,

$\alpha^2-\alpha\beta+\alpha^2+\alpha\beta+\alpha^2-\beta^2 = 11$

or, $3(-2)^2-\beta^2 = 11$ or, $\beta = \pm 1$.

The roots of the equ^n are when $\alpha = -2, \beta = 1$

$\therefore$ The roots of the equ^n are $(-2-1), -2, (-2+1)$ i.e $-3, -2, -1$.

The roots of the equ^n are when $\alpha = -2, \beta = -1$,

" " " " " $(-2+1), -2, (-2-1)$ i.e $-1, -2, -3$ A

(ii) $4x^4-4x^3-21x^2+11x+10 = 0$

A:– The given equ^n are $4x^4-4x^3-21x^2+11x+10 = 0$ (1)

Since the roots are in A.P, then roots are $\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$.

From the relation between roots and co-efficients,

$\alpha+\beta+\gamma+\delta =$

$4\alpha = 1$ (2)

$(\alpha-3\beta+\alpha+3\beta)(\alpha+\beta+\alpha-\beta)+(\alpha+\beta)(\alpha-\beta)+(\alpha-3\beta)(\alpha+3\beta) = -\frac{21}{4}$ (3)

$(\alpha+\beta)(\alpha-\beta)(\alpha+3\beta)(\alpha-3\beta) = \frac{10}{4}$ (4)

from (2) $\alpha = \frac{1}{4}$

From (3) $6\alpha^2 - 10\beta^2 = -\frac{21}{4}$ or, $\beta = \pm\frac{3}{4}$

when $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$ then roots are $-2, -\frac{1}{2}, 1, \frac{5}{2}$,

when $\alpha = \frac{1}{4}$, $\beta = -\frac{3}{4}$, then roots are,

$\frac{5}{2}, 1, -\frac{1}{2}, -2$ Ans

(iii) $4x^4 + 20x^3 + 35x^2 + 25x + 6 = 0$

Ans:- The given equ^n is $4x^4 + 20x^3 + 35x^2 + 25x + 6 = 0$ (1)

Since the roots are in A.P then $\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$.

From the relation between roots and co-efficients we have,

$4\alpha = -\frac{20}{4}$ (2)

$(\alpha-3\beta+\alpha+3\beta)(\alpha+\beta+\alpha-\beta)+(\alpha-\beta)(\alpha+\beta)+(\alpha-3\beta)(\alpha+3\beta) = \frac{35}{4}$ (3)

$(\alpha+3\beta)(\alpha-3\beta)(\alpha+\beta+\alpha-\beta)+(\alpha+\beta)(\alpha-\beta)(\alpha+3\beta+\alpha-3\beta) = -\frac{25}{4}$ (4)

$(\alpha-\beta)(\alpha+\beta)(\alpha-3\beta)(\alpha+3\beta) = \frac{6}{4}$ (5)

From, $\alpha = -\frac{5}{4}$

$6(-\frac{5}{4})^2 - 10\beta^2 = \frac{35}{4}$ or, $\beta = \frac{1}{4}$

∴ the roots of the equ^n are $-2, -\frac{3}{2}, -1, -\frac{1}{2}$ Ans.

4. (i) Solve the equ^{n}s given that the roots are in geometric progressi

$3x^3 - 26x^2 + 52x - 24 = 0$.

Ans:- The given equ^n is $3x^3 - 26x^2 + 52x - 24 = 0$ (1)

If the roots are in G.P, then we say that, the roots are α, β, γ,

$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\alpha}$ ~~Since~~

$\alpha\beta = \gamma^2$ (2)

From the relation between roots and co-efficient we get,

$\alpha+\beta+\gamma = \frac{26}{3}$ (3) $\alpha\beta\gamma = -\frac{24}{3}$ (5)

$\alpha\beta + \gamma\alpha + \beta\gamma = \frac{52}{3}$ (4)

From (2), $\beta^3 = 8$ ∴ $\beta = 2$

From (2) $\alpha\gamma = 4$ (6)

From (4) $\beta(\gamma+\alpha) + 4 = \frac{52}{3}$

or, $\alpha + \gamma = \frac{20}{3}$

$(\alpha-\gamma)^2 = (\alpha+\gamma)^2 - 4\alpha\gamma$

$(\alpha-\gamma) = 16/3$

$\alpha + \gamma = \frac{20}{3}$

$\alpha - \gamma = \frac{16}{3}$

$2\alpha = \frac{36}{3}$ ∴ The roots of the equⁿ are $6, 2, \frac{2}{3}$ Ans.

$\alpha = 6$

Q. (ii) $x^4 - 5x^3 - 30x^2 + 40x + 64 = 0$ (1)

A:- The given equⁿ is $x^4 - 5x^3 - 30x^2 + 40x + 64 = 0$ (1). If the roots are in

Let $\alpha, \beta, \gamma, \delta$, are the roots of the equⁿ (1)

G.P, $\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta}$

$\alpha\delta = \beta\gamma$ (2)

From the relation between roots and co-efficients,

$\alpha + \beta + \gamma + \delta = 5$ (3)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = -30$ (4)

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -40$ (5)

$\alpha\beta\gamma\delta = 64$ (6)

From (2) we get and (6) | From (5) we get,

$(\alpha\delta)^2 = 64$ | $\alpha\delta = \beta\gamma = -8$

or, $\alpha\delta = \pm 8$

From (4) we get,

$(\alpha + \delta)(\gamma + \beta) = -14$

Now, $(\alpha + \delta), (\gamma + \beta)$ are the roots of the equⁿ,

$t^2 - (\alpha + \beta + \gamma + \delta)t + (\alpha + \delta)(\beta + \gamma) = 0$

or $t^2 - 5t - 14 = 0$

$t = 7, -2$

We take $\alpha + \delta = 7$, $\beta + \gamma = -2$

$\{t^2 - (\alpha + \delta)t + \alpha\delta\}\{t^2 - (\beta + \gamma)t + \beta\gamma\} = 0$

or, $t^2 - 7t - 8 = 0$ | $t^2 + 2t - 8 = 0$

$t = 8, -1$ | $t = 2, -4$

∴ The roots of the equⁿ are $-1, 8, 2, -4$ Ans.

Q. (iii) $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$

A:- The given equⁿ is $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$ (1)

If the roots are in G.P then the roots are

$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta}$

$\alpha\delta = \beta\gamma$ (2)

From the relation between roots and co-efficients,

$\alpha + \beta + \gamma + \delta = -15$ (3)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = 70$ (4)

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = -120$ (5)

$\alpha\beta\gamma\delta = 64$ (6)

$\beta\gamma = \alpha\delta = \pm 8$

From (5) we get, $\alpha\delta = \beta\gamma = 8$

$(\alpha+\delta)(\beta+\gamma) = 54$ (7)

Now, $(\alpha+\delta)$ and $(\beta+\gamma)$ are the roots of the equn,

$t^2-(\alpha+\beta+\gamma+\delta)t+(\alpha+\delta)(\beta+\gamma)=0$

or, $t^2-15t+54=0$

$t=-9,-6$

We take, $\alpha+\delta=-9$, $\beta+\gamma=-6$.

$\{t^2-(\alpha+\delta)t+\alpha\delta\}\{t^2-(\beta+\gamma)t+\beta\gamma\}=0$

or, $t^2+9t+8=0$ | $t^2+6t+8=0$

$t=-1,-8$ | $t=-4,-2$

$\therefore$ Roots of the equn are $-1,-8,-4,-2$

(iv) $3x^4+20x^3-70x^2-60x+27=0$

Given equn is $3x^4+20x^3-70x^2-60x+27=0$ (1)

Since the equn is $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$ or $\alpha\delta=\beta\gamma$ (2).

from the relation between roots and co-efficients, we get,

$\alpha+\beta+\gamma+\delta=-\frac{20}{3}$ (3)

$(\alpha+\delta)(\gamma+\beta)+\alpha\beta+\gamma\delta=-\frac{70}{3}$ (4)

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=\frac{60}{3}$ (5)

$\alpha\beta\gamma\delta=\frac{27}{3}$ (6)

from (2) and (6) $\beta\gamma=\pm 3$

From (5) we get, $\alpha\delta=\beta\gamma=-3$.

Now, $(\alpha+\delta),(\beta+\gamma)$ are the roots of the equn,

$t^2-(\alpha+\delta+\beta+\gamma)t+(\alpha+\delta)(\beta+\gamma)=0$

or, $t^2+\frac{20}{3}t-\frac{52}{3}=0$ or, $3t^2+20t-52=0$

or, $3t^2+26t-6t-52=0$ or, $t(3t+26)-2(3t+26)=0$

or, $(3t+26)(t-2)=0$, $t=2,-26/3$.

We take, $\alpha+\delta=-26/3$, $\beta+\gamma=2$,

$\{t^2-(\alpha+\delta)t+\alpha\delta\}\{t^2-(\beta+\gamma)t+\beta\gamma\}=0$

or, $t^2+\frac{26}{3}t-3=0$ | $t^2-2t-3=0$

or, $t=-9,\frac{1}{3}$ | $t=3,-1$.

$\therefore$ Roots are $-9,\frac{1}{3},3,-1$

$- 3(\gamma+\delta) - 1(\alpha+\beta) = -6$

or, $(\alpha+\beta) = 3(\gamma+\delta) + 6$

from (3), $\alpha+\beta+\gamma+\delta = -10$

or, $3(\gamma+\delta)+6+\gamma+\delta = -10$

or, $\gamma+\delta = -4$

$\alpha+\beta = 3(-4)+6 = -6$

$\{t^2 - (\alpha+\beta)t + \alpha\beta\}\{t^2 - (\gamma+\delta)t + \gamma\delta\} = 0$

or, $t^2+6t+3 = 0$	$t^2+4t-1 = 0$
$t = \dfrac{-6 \pm \sqrt{36-4.3.1}}{2}$	$t = -2 \pm \sqrt{3}$,
$= -3 \pm \sqrt{6}$	

$\therefore$ Roots of the equ^n $-3\pm\sqrt{6}$, $-2\pm\sqrt{3}$, A,

(1) Solve the equ^{n}s given that the sum of two of the roots is equal to the sum of the other two.

$x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$,

A:- $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ (1)

Given that $\alpha+\beta = \gamma+\delta$ (2)

$\alpha+\beta+\gamma+\delta = -2$ (3)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = -21$ (4)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -22$ (5)

$\alpha\beta\gamma\delta = 40$ (6)

From (2), $\alpha+\beta+\gamma+\delta = -2$

or, $2(\alpha+\beta) = -2$

or, $\alpha+\beta = -1$

$\gamma+\delta = -1$

From (5)

$(\alpha+\beta)(\gamma\delta + \alpha\beta) = -22$

or, $\alpha\beta + \gamma\delta = -22$ (7)

Now, $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n,

$t^2 - (\alpha\beta + \gamma\delta)t + \alpha\beta\gamma\delta = 0$

or, $t^2 + 22t + 40 = 0$

or, $t^2 + 20t + 2t + 40 = 0$

or, $t(t+20) + 2(t+20) = 0$

$t = -20, -2$,

$\alpha\beta = -20$, $\gamma\delta = -2$

$\{t^2 - (\alpha+\beta)t + \alpha\beta\}\{t^2 - (\gamma+\delta)t + \gamma\delta\} = 0$

or, $t^2+t-20 = 0$	$t^2+t-2 = 0$
$t = 4, -5$	$t = 1, -2$

$\therefore$ Roots are $4, -5, 1, -2$. A

(ii) $x^4 - 8x^3 + 21x^2 - 20x + 6 = 0$,

Ans:- Given equ^n is $x^4 - 8x^3 + 21x^2 - 20x + 6 = 0$ (1)

$\alpha + \beta = \gamma + \delta$ (2)

From the relation between roots and co-efficients we get,

$\alpha + \beta + \gamma + \delta = 8$ (3)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = 21$ (4)

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 20$ (5)

$\alpha\beta\gamma\delta = 6$ (6)

From (2) and (3), $\alpha + \beta = \gamma + \delta = 4$

$2(\alpha + \beta) = 8$ $\alpha\beta + \gamma\delta = 21 - 16 = 5$

$\alpha + \beta = 4$ (7)

Now, $\alpha\beta$, $\gamma\delta$ are the roots of the equ^n

$t^2 - (\alpha\beta + \gamma\delta)t + \alpha\beta\gamma\delta = 0$

or, $t^2 - 5t + 6 = 0$

or, $(t-2)(t-3) = 0$, $t = 2, 3$

We take $\alpha\beta = 2$ and $\gamma\delta = 3$

$t^2 - (\alpha + \beta)t + \alpha\beta = 0$

or, $t^2 - 4t + 2 = 0$

$t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

$t^2 - (\gamma + \delta)t + \gamma\delta = 0$

or, $t^2 - 4t + 3 = 0$

$t = 3, 1$

$\therefore$ Roots of the equ^n are $3, 1, 2 \pm \sqrt{2}$. Ans

7. Solve the equ^{n}s, given that the product of two of roots is equal to the product of the other two.

(i) $x^4 + 3x^3 - 4x^2 - 9x + 9 = 0$,

Ans:- $x^4 + 3x^3 - 4x^2 - 9x + 9 = 0$ (1)

Let roots of the equ^n are $\alpha, \beta, \gamma, \delta$,

Given that $\alpha\beta = \gamma\delta$ (2)

From the relation between roots and co-efficients,

$\alpha + \beta + \gamma + \delta = -3$ (3)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = -4$ (4)

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 9$ (5)

$\alpha\beta\gamma\delta = 9$ (6)

From (2) and (6) we get,

$\alpha\beta = \pm 3$

From (5) we get, $\alpha\beta = \gamma\delta = -3$,

From (4), $(\alpha + \beta)(\gamma + \delta) = -4 + 6 = 2$

$t^2 - (\alpha + \beta + \gamma + \delta)t + (\alpha + \beta)(\gamma + \delta) = 0$

or, $t^2 + 3t + 2 = 0$

or, $(t+2)(t+1)=0$

$t=-2,-1$

We take $\alpha+\beta=-2,\quad \gamma+\delta=-1$

$t^2-(\alpha+\beta)t+\alpha\beta=0 \quad | \quad t^2-(\gamma+\delta)t+\gamma\delta=0$

or, $t^2+2t-3=0 \quad | \quad$ or, $t^2+t-3=0$

$t=1,-3 \quad | \quad t=-1\pm\sqrt{3}/2$

$\therefore$ Roots of the equⁿ are $1,-3,\ \dfrac{-1\pm\sqrt{3}}{2}$ Ans.

7.(ii) $2x^4+x^3+2x^2+3x+18=0$

A:- Given equⁿ is $2x^4+x^3+2x^2+3x+18=0$ (1)

Let the roots are $\alpha,\beta,\gamma,\delta$

Given that $\alpha\beta=\gamma\delta$ (2)

From the relation between roots and co-efficients we get,

$\alpha+\beta+\gamma+\delta=-\frac{1}{2}$ (3)

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=1$ (4)

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-3/2$ (5)

$\alpha\beta\gamma\delta=18/2=9$ (6)

From (2) and (6) $\quad$ From (5) we get

$\alpha\beta=\pm3 \quad \alpha+\beta+\gamma+\delta=-1/2,\quad \alpha\beta=\gamma\delta=3.$

$\therefore (\alpha+\beta)(\gamma+\delta)=-5$

$t^2-(\alpha+\beta+\gamma+\delta)t+(\alpha+\beta)(\gamma+\delta)=0$

or, $t^2+\frac{1}{2}t-5=0$

or, $2t^2+t-10=0$

$t=2,-\frac{5}{2}$, We take $\alpha+\beta=2,\ \gamma+\delta=-5/2$

$t^2-2t+3=0 \quad | \quad t^2+\frac{5}{2}t+3=0$

$t=1\pm i\sqrt{2} \quad | \quad t=\dfrac{-5\pm\sqrt{23}\,i}{4}$

$\therefore$ The roots of the equⁿ are $1\pm i\sqrt{2},\ \dfrac{-5\pm\sqrt{23}\,i}{4}$ Ans

8.(i) Solve the equⁿs given that the ratio of the roots is equal to

$x^4-12x^3+47x^2-72x+36=0$

A:- Given equⁿ is $x^4-12x^3+47x^2-72x+36=0$ (1)

Let $\alpha,\beta,\gamma,\delta$ are the roots of the equⁿ, $\alpha\beta=\gamma\delta$ (2)

From the relation

$\alpha+\beta+\gamma+\delta=12$ (3)

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=47$ (4)

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=72$ (5)

$\alpha\beta\gamma\delta=36$ (6)

$\alpha\beta=\pm6$

From (6) $\alpha\beta=\gamma\delta=6,$

$(\alpha+\beta)(\gamma+\delta)=35$ (7)

$t^2 - (\alpha+\beta+\gamma+\delta)t + (\alpha+\beta)(\gamma+\delta) = 0$

or, $t^2 - 12t + 35 = 0$

or, $t^2 - 7t - 5t + 35 = 0$ $\qquad t = 5, 7$,

or, $t(t-7) - 5(t-7) = 0$

$\{t^2 - (\alpha+\beta)t + \alpha\beta\}$ $\qquad \{t^2 - (\gamma+\delta)t + \gamma\delta\} = 0$

or, $t^2 - 5t + 6 = 0$ $\qquad t^2 - 7t + 6 = 0$

or, $t^2 - 3t - 2t + 6 = 0$ $\qquad$ or, $t(t-6) - 1(t-6) = 0$

$t = 3, 2$ $\qquad t = 1, 6$

$\therefore$ Roots of the equ^n are $\quad 3, 2, 6, 1$. $\quad$ Ans

(ii) $x^4 + 2x^3 - 18x^2 + 6x + 9 = 0$

Ans:- Given that $x^4 + 2x^3 - 18x^2 + 6x + 9 = 0$ (1)

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n, $\alpha\beta = \gamma\delta$ (2)

From the relation between roots and co-efficients,

$\alpha+\beta+\gamma+\delta = -2$ (3)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = -18$ (4)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -6/1$ (5)

$\alpha\beta\gamma\delta = 9$ (6)

$\alpha\beta = \pm 3$

From (5) $\quad \alpha\beta = \gamma\delta = 3$,

$(\alpha+\beta)(\gamma+\delta) + 3 + 3 = -18$ $\qquad$ or, $(\alpha+\beta)(\gamma+\delta) = -24$

$t^2 - (\alpha+\beta+\gamma+\delta)t + (\alpha+\beta)(\gamma+\delta) = 0$

or, $t^2 + 2t - 24 = 0$

or, $(t+6)(t-4) = 0$ $\qquad t = 4, -6$

$\alpha+\beta = 4$, $\qquad \gamma+\delta = -6$

$t^2 - (\alpha+\beta)t + \alpha\beta = 0$ $\qquad t^2 - (\gamma+\delta)t + \gamma\delta = 0$

$\therefore t = 3, 1$ $\qquad t = -3 \pm \sqrt{6}$

$\therefore$ Roots of the equ^n are $\quad 3, 1, -3 \pm \sqrt{6}$ $\quad$ Ans

(iii) $2x^4 + 3x^3 - 19x^2 + 6x + 8 = 0$.

Ans:- Given equ^n is $2x^4 + 3x^3 - 19x^2 + 6x + 8 = 0$ (1)

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n (1),

Given that $\alpha\beta = \gamma\delta$ (2)

$\alpha+\beta+\gamma+\delta = -3/2$ (3)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = -19/2$ (4)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -6/2$ (5)

$\alpha\beta\gamma\delta = 8/2$ (6)

from (2) and (6) we get $\qquad$ from (5) we get

$\alpha\beta = \pm 2$ $\qquad \alpha\beta = \gamma\delta = 2$

From (4), $(\alpha+\beta)(\gamma+\delta) = -27/2$,

$t^2 + \frac{3}{2}t - \frac{27}{2} = 0$ or, $2t^2 + 3t - 27 = 0$,

$t = 3, -9/2$,

$\therefore \alpha+\beta = 3$ and $\gamma+\delta = -9/2$

$t^2 - 3t + 2 = 0$ $\quad$ $t^2 + \frac{9}{2}t + 2 = 0$

or, $t = 2, 1$, $\quad$ $t = -4, -\frac{1}{2}$,

$\therefore$ The roots of the equ^n $1, 2, -4, -\frac{1}{2}$. Ans

9. (i) Determine k and solve the equ^n if the roots are in A.P.

$8x^3 - 12x^2 - kx + 3 = 0$.

Ans:- Let the roots of the equ^n are $\alpha-\beta, \alpha, \alpha+\beta$. we get, from the relation between roots and co-efficients

$3\alpha = \frac{12}{8}$ or, $\alpha = \frac{1}{2}$ (3)

$\alpha(\alpha-\beta) + \alpha(\alpha+\beta) + (\alpha+\beta)(\alpha-\beta) = -\frac{k}{8}$

or, $3\alpha^2 - \beta^2 = -\frac{k}{8}$

or, $k = 8\beta^2 - 6$

$\alpha(\alpha-\beta)(\alpha+\beta) = -\frac{3}{8}$

or, $\frac{1}{2}\left(\frac{1}{4} - \beta^2\right) = -\frac{3}{8}$ $\quad$ $\therefore k = 8 - 6 = 2$,

or, $\beta^2 = 1$ or, $\beta = \pm 1$

$\therefore$ Roots of the equ^n are $(\frac{1}{2}-1), \frac{1}{2}, (\frac{1}{2}+1)$ i.e $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$,

$\alpha = \frac{1}{2}, \beta = -1$, roots are $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$. Ans

(ii) $x^4 - 8x^3 + kx^2 + 8x - 15 = 0$. --- (1)

Ans:- Let the roots of the equ^n are $\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$. From the relation between roots and coefficient we get,

$\alpha = 2$, (2)

$(\alpha+3\beta+\alpha-3\beta)(\alpha+\beta+\alpha-\beta) + (\alpha+3\beta)(\alpha-3\beta) + (\alpha+\beta)(\alpha-\beta) = k$

or, $k = 24 - 10\beta^2$ (3)

$(\alpha+\beta)(\alpha-\beta)(\alpha+3\beta+\alpha-3\beta) + (\alpha+3\beta)(\alpha-3\beta)(\alpha+\beta+\alpha-\beta) = -8$

or, $(\alpha^2-\beta^2)2\alpha + (\alpha^2-9\beta^2)2\alpha = -8$

or, $(2\alpha^2 - 10\beta^2) = -2$

or, $\beta = \pm 1$

We take $\alpha = 2, \beta = 1$. $\quad$ $k = 14$.

$\therefore$ Then the roots of the equ^n are $-1, 1, 3, 5$,

when $\alpha = 2, \beta = -1$, $\quad$ $k = 14$,

Then the roots of the equ^n are $5, 3, 1, -1$. Ans

10. (i) Find the relation among the co-efficient of the equation $ax^3 + 3bx^2 + 3cx + d = 0$

If the roots lie in (i) A.P (ii) G.P (iii) H.P.

Ans:- Given equⁿ is $ax^3 + 3bx^2 + 3cx + d = 0$ (1)

Since the roots of (1) are in A.P.

Let the roots are $\alpha - \beta, \alpha, \alpha + \beta$.

From the relation between roots and co-effi^nt

$$\alpha = -\frac{b}{a} \quad (2)$$

$$(\alpha - \beta)\alpha + \alpha(\alpha + \beta) + \alpha^2 - \beta^2 = \frac{3c}{a}$$

or, $$\beta^2 = \frac{3}{a^2}(b^2 - ac).$$

$$(\alpha - \beta)(\alpha + \beta)\alpha = -\frac{d}{a}$$

or, $$(\alpha^2 - \beta^2)\alpha = -\frac{d}{a}$$

or, $$\left\{\left(-\frac{b}{a}\right)^2 - \left(\frac{3b^2 - 3ac}{a^2}\right)\right\}\left(-\frac{b}{a}\right) = -\frac{d}{a}$$

or, $$\left\{\frac{b^2 - 3b^2 + 3ac}{a^2}\right\} = \frac{d}{b}$$

or, $$3abc - 2b^3 = a^2 d \quad \text{Ans.}$$

(ii) Given equⁿ is $ax^3 + 3bx^2 + 3cx + d = 0$ (1)

Since the roots of (1) are in G.P.

Let the roots are $\frac{\alpha}{\beta}, \alpha, \alpha\beta$.

From the relation between roots and co-efficients,

$$\frac{\alpha}{\beta} + \alpha + \alpha\beta = -\frac{3b}{a}$$

or, $$\alpha\left(\frac{1}{\beta} + 1 + \beta\right) = -\frac{3b}{a} \quad (2)$$

$$\frac{\alpha}{\beta}\cdot\alpha + \alpha\cdot\alpha\beta + \frac{\alpha}{\beta}\cdot\alpha\beta = \frac{3c}{a}$$

or, $$\alpha^2\left(\frac{1}{\beta} + 1 + \beta\right) = -\frac{3b}{a} \quad (3)$$

$$\frac{\alpha}{\beta}\cdot\alpha\cdot\alpha\beta = -\frac{d}{a} \quad \text{or,} \quad \alpha^3 = -\frac{d}{a} \quad (4)$$

[(3) ÷ (2) we get]

$$\alpha = -\frac{c}{b}$$

from (4), $$\left[-\frac{c}{b}\right]^3 = -\frac{d}{a}$$

or, $ac^3 = bd^3$ which is the required equⁿ [A

(iii) The given equⁿ is $ax^3 + 3bx^2 + 3cx + d = 0$ (1)

Let α, β, γ are the roots of the equⁿ (1).

Since, α, β, γ are in H.P, so $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P.

Now $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of the equⁿ,

$$dx^3 + 3cx^2 + 3bx + a = 0 \quad (2)$$

∴ Roots of the equⁿ (2) are in A.P,

Let the roots of the equⁿ (2) are $(\alpha' - \beta'), \alpha', (\alpha' + \beta')$.

From the relation between roots and co-efficients,

$$\alpha' = -\frac{c}{d} \quad (3)$$

12. (i) If the equⁿ $x^3+px^2+qx+r=0$ has a root $\alpha+i\alpha$ where p, q, r and α are real, prove that $(p^2-2q)(q^2-2pr)=r^2$.

Hence solve the equⁿs (i) $x^3-x^2-4x+24=0$

(ii) $x^3-7x^2+20x+24=0$.

A:- Given equⁿ is

$$x^3+px^2+qx+r=0 \quad (1)$$

since p, q, r are real and $\alpha+i\alpha$ is root of the equⁿ (1) then, $\alpha-i\alpha$ is also another root of (1).

Let the other root is β.

From the relation between roots and co-efficient we get,

$$\alpha+i\alpha+\alpha-i\alpha+\beta=-p \quad \text{or,} \quad 2\alpha+\beta=-p \quad (2).$$

$$(\alpha+i\alpha)(\alpha-i\alpha)+(\alpha+i\alpha)\beta+(\alpha-i\alpha)\beta=q$$

or, $$2\alpha^2+2\alpha\beta=q \quad (3)$$

$$(\alpha-i\alpha)(\alpha+i\alpha)\beta=-r$$

or, $$2\alpha^2\beta=-r \quad (4)$$

L.H.S

$$(p^2-2q)(q^2-2pr)$$

$$=\{(2\alpha+\beta)^2-2(2\alpha^2+2\alpha\beta)\}\{(2\alpha^2+2\alpha\beta)^2-2(2\alpha+\beta).2\alpha^2\beta\}$$

$$=\{4\alpha^2+\beta^2+4\alpha\beta-4\alpha^2-4\alpha\beta\}\{4\alpha^4+4\alpha^2\beta^2+8\alpha^3\beta-8\alpha^3\beta-4\alpha^2\beta^2\}$$

$$=\beta^2.4\alpha^4=(2\alpha^2\beta)^2=r^2 \quad = \text{R.H.S.}$$

$\therefore$ L.H.S = R.H.S [proved]

(i) $x^3-x^2-4x+24=0$ (5)

comparing the equⁿ (5) with the equⁿ (1) we get,

$p=1$, $q=-4$, $r=24$.

Now, $(p^2-2q)(q^2-2pr)=r^2$,

$=(1+8)(16+48)$

$\therefore$ the roots of the equⁿ (5) is of the form $(\alpha+i\alpha), (\alpha-i\alpha), \beta$.

Relation between roots and co-efficients we get,

$$2\alpha+\beta=1 \quad (6)$$

$$2\alpha^2+2\alpha\beta=-4 \quad (7)$$

$$2\alpha^2+\alpha\beta=\alpha$$

$$-\ \cancel{2}\alpha^2+2\alpha\beta=-4$$

$$-\alpha\beta=4+\alpha$$

or, $$\beta=-\left(\frac{4+\alpha}{\alpha}\right)$$

$$2\alpha^2\beta=-24 \quad (8)$$

$$2\alpha^2\beta=-24$$

or, $$-\alpha^2\left(\frac{4+\alpha}{\alpha}\right)=-12$$

or, $$(\alpha+6)(\alpha-2)=0$$

$$\alpha=2, -6.$$

$$\beta=-3.$$

when, $\alpha=2$, $\beta=-\frac{1}{3}$.

when, $\alpha=-6$, $\beta=-3$ then,

when $\alpha=2$, $\beta=-\frac{1}{3}$,

$2\alpha+\beta=1$; $\alpha=-6$;

when,

$2\alpha+\beta=-37/3 \neq 1$.

$\therefore$ The roots of the equⁿ are $(2+2i), (2-2i), -3$

A

Now, $p^2 - 2q$ $\qquad r^2 - 2qs$

$= (4)^2 - 2(8) = 0$ $\quad = \{(-24)^2 - 2 \cdot 8 \cdot 36\} = 0$,

$\therefore$ the roots of the equⁿ ⑤ is of the form $(\alpha \pm i\alpha), (\beta \pm i\beta)$.

Relatⁿ between roots and co-efficient we get,

$2(\alpha + \beta) = -4$ ⑥ $\qquad 2(\alpha^2 + 2\alpha\beta + \beta^2) = 8$ ⑦

$4\alpha\beta(\alpha+\beta) = 24$ ⑧ $\qquad \alpha^2\beta^2 = 9$

$\alpha\beta = \pm 3$

From ⑧ we get $\alpha\beta = 3$.

From ⑥ $\quad \alpha+\beta = -2$. or, $(\alpha+\beta)^2 = 4$.

or, $(\alpha - \beta)^2 + 4\alpha\beta = 4$ or, $(\alpha-\beta)^2 = 8i^2 = (2\sqrt{2}i)^2$

or, $\alpha - \beta = 2\sqrt{2}i$

$\alpha + \beta = -2$

$\alpha = (\sqrt{2}i - 1)$ $\qquad \beta = (\sqrt{2}i - 1 - 2\sqrt{2}i)$ Ans

14.(i) Solve the equⁿs $x^4 + 2x^3 + 5x^2 + 4x + 4 = 0$ given that each has two distinct pairs of equal roots.

A:- Given equⁿ is $x^4 + 2x^3 + 5x^2 + 4x + 4 = 0$ ①

Let the roots of the equⁿ ① are $\alpha, \alpha, \beta, \beta$,

From the relatⁿ between roots and co-efficients we get,

$2(\alpha+\beta) = -2$ or, $\alpha+\beta = -1$ ②.

$(\alpha+\alpha)(\beta+\beta) + \alpha^2 + \beta^2 = 5$ or, $\alpha^2 + \beta^2 + 4\alpha\beta = 5$ ③

$\alpha^2(\beta+\beta) + \beta^2(\alpha+\alpha) = -4$ or, $2\alpha^2\beta + 2\beta^2\alpha = -4$

or, $\alpha^2\beta + \beta^2\alpha = -2$ ④

$\alpha^2\beta^2 = 4$ ⑤

From ② & ④ $\quad \alpha\beta(\alpha+\beta) = -2$

or, $\alpha\beta = 2$

Now, α, β are the roots of the equⁿ, $t^2 - (\alpha+\beta)t + \alpha\beta = 0$

or, $t^2 + t + 2 = 0$ or, $t = \dfrac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 1}}{2} = \dfrac{-1 \pm \sqrt{7}i}{2}$

We take, $\alpha = -\dfrac{1 + \sqrt{7}i}{2}$, $\beta = -\dfrac{1 - \sqrt{7}i}{2}$.

$\therefore$ The roots of the equⁿ are $\dfrac{-1 \pm \sqrt{7}i}{2}$, $\dfrac{-1 \pm i\sqrt{7}}{2}$ Ans

(ii) Given equⁿ is $4x^4 + 20x^3 + 13x^2 - 30x + 9 = 0$ ①

Let the roots of the equⁿ ① are $\alpha, \alpha, \beta, \beta$.

From the relation between roots and co-efficients we get,

$2\alpha + 2\beta = -\dfrac{20}{4}$ or, $\alpha+\beta = -5/2$ ②

$(\alpha+\alpha)(\beta+\beta) + \alpha^2 + \beta^2 = \dfrac{13}{4}$ or, $4\alpha\beta + \alpha^2 + \beta^2 = \dfrac{13}{4}$ ③

$\alpha^2(\beta+\beta) + \beta^2(\alpha+\alpha) = \dfrac{30}{4}$ or, $2\alpha^2\beta + 2\beta^2\alpha = \dfrac{30}{4}$

or, $\alpha^2\beta + \beta^2\alpha = \dfrac{30}{8}$ ④

$\alpha^2\beta^2 = 9$ ⑧

15. (a) If the product of two roots of the equⁿ $x^4 + px^3 + qx^2 + rx + s = 0$ be equal to the product of the other two, prove that $r^2 = p^2 s$. If $p \neq 0$, show that the equⁿ can be solved by the substitution

$$x + \frac{r}{px} = t$$

A: Given equⁿ is → প্রথম Part-টি আগে করতে হবে।

$$x^4 + px^3 + qx^2 + rx + s = 0 \quad (1)$$

or, $\left(x^2 + \frac{s}{x^2}\right) + \left(px + \frac{r}{x}\right) + q = 0$

or, $\left(x^2 + \frac{r^2}{p^2x^2}\right) + p\left(x + \frac{r}{px}\right) + q = 0 \quad \left[\because s = \frac{r^2}{p^2}\right]$

or, $\left(x + \frac{r}{px}\right)^2 - 2 \cdot x \cdot \frac{r}{px} + p\left(x + \frac{r}{px}\right) + q = 0 \quad (2)$

By the substitution $x + \frac{r}{px} = t$, equⁿ (2) becomes,

$$t^2 - \frac{2r}{p} + pt + q = 0$$

or, $t^2 + pt + \left(q - \frac{2r}{p}\right) = 0 \quad (3)$

This is a quadratic equⁿ in t,

∴ Equⁿ (3) gives two values of t.

Since $x + \frac{r}{px} = t$,

∴ Each values of t we get two values of x,

∴ The given equⁿ can be solved by the substitution $x + \frac{r}{px} = t$. [Proved]

(b) The given equⁿ is $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0 \quad (1)$

The equⁿ (1) can be written as,

$$x^2 - 12x + 47 - \frac{72}{x} + \frac{36}{x^2} = 0$$

or, $\left(x^2 + \frac{36}{x^2}\right) - \left(12x + \frac{72}{x}\right) + 47 = 0$

or, $\left\{\left(x + \frac{6}{x}\right)^2 - 2 \cdot x \cdot \frac{6}{x}\right\} - 12\left(x + \frac{6}{x}\right) + 47 = 0 \quad \left[\text{use substitution } x + \frac{36}{x} = t\right]$

or, $t^2 - 12t - 12 + 47 = 0$

or, $t^2 - 12t + 35 = 0$

$t = 7, 5$

when $t = 7$ then $x + \frac{6}{x} = 7$ or, $x = 6, 1$.

when $t = 5$, then, $x + \frac{6}{x} = 5$ or, $x^2 - 5x + 6 = 0$

or, $x = 3, 2$.

∴ Roots of the given equⁿ are $6, 1, 3, 2$. Ans

15 (c) (i) The given equⁿ is $x^4 - 5x^3 - 30x^2 + 40x + 64 = 0 \quad (1)$

The equⁿ (1) can be written as,

$$x^2 - 5x - 30 + \frac{40}{x} + \frac{64}{x^2} = 0$$

or, $\left(x^2 + \frac{64}{x^2}\right) - 5\left(x - \frac{8}{x}\right) - 30 = 0$

or, $\left\{\left(x - \frac{8}{x}\right)^2 + 2 \cdot x \cdot \frac{8}{x}\right\} - 5\left(x - \frac{8}{x}\right) - 30 = 0$

or, $\left(x - \frac{8}{x}\right)^2 - 5\left(x - \frac{8}{x}\right) - 14 = 0 \quad \left[\text{use substitution } x - \frac{8}{x} = t\right]$

$t^2 - 5t - 14 = 0$

when t = 7, $x - \frac{8}{x} = 7$ or, $x = 8, -1$.

when t = −2 $x - \frac{8}{x} = -2$ or, $x = 2, -4$.

∴ The roots of the eqn are $-1, 2, -4, 8$. Ans.

The given eqn is $3x^4 + 20x^3 - 70x^2 - 60x + 27 = 0$ (1)

(ii) The given eqn (1) can be written as,

$$3x^2 + 20x - 70 - \frac{60}{x} + \frac{27}{x^2} = 0$$

or, $3\{(x - \frac{3}{x})^2 + 2.x.\frac{3}{x}\} + 20(x - \frac{3}{x}) - 70 = 0$

or, $3(x - \frac{3}{x})^2 + 20(x - \frac{3}{x}) - 52 = 0$ [use substitution $x - \frac{3}{x} = t$]

or, $3t^2 + 20t - 52 = 0$

or, $t = 2, -\frac{26}{3}$,

$x - \frac{3}{x} = 2$ $x - \frac{3}{x} = -\frac{26}{3}$

or, $x = 3, -1$ or, $x = -9, \frac{1}{3}$.

∴ The roots of the eqn are $\frac{1}{3}, -1, 3, -9$. Ans.

16. The given eqn is $x^3 + px^2 + qx + r = 0$ (1). From the relation between roots & co-efficient we get,

$\Sigma\alpha = -p$ (2) $\Sigma\alpha\beta = q$, (3) $\alpha\beta\gamma = -r$ (4).

(i) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ | $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca)$

$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\,\Sigma\alpha$

$= (q)^2 - 2(-r)(-p)$ Ans.

$= q^2 - 2pr$

(ii) $\Sigma\alpha^3\beta^3 = \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$

$= \Sigma\alpha\beta\,(\Sigma(\alpha\beta)^2 - \alpha\beta\gamma\,\Sigma\alpha) + 3(\alpha\beta\gamma)^2$

$= q\{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha) - \alpha\beta\gamma\,\Sigma\alpha\} + 3(\alpha\beta\gamma)^2$

$= q(q^2 - 2pr - pr) + 3(-r)^2$ Ans.

$= q^3 - 3pqr + 3r^2$

(iii) $\Sigma(\beta+\gamma-\alpha)^3 = \Sigma(-p-\alpha-\alpha)^3$ [∵ $\alpha+\beta+\gamma = -p$, $\beta+\gamma = -p-\alpha$]

$= -\Sigma(p+2\alpha)^3$

$= -\Sigma(p^3 + 6p^2\alpha + 12p\alpha^2 + 8\alpha^3)$

$= -[\Sigma p^3 + 6p^2\Sigma\alpha + 12p\,\Sigma\alpha^2 + 8\,\Sigma\alpha^3]$

$= -[3p^3 + 6p^2(-p) + 12p\{(\Sigma\alpha)^2 - 2\Sigma\alpha\beta\} + 8\{\Sigma\alpha^2\Sigma\alpha - \Sigma\alpha^2\beta\}]$

$= -[3p^3 - 6p^3 + 12p\{(-p)^2 - 2q\} + 8\{\cancel{\Sigma\alpha}(\Sigma\alpha)^2 - 2\Sigma\alpha\beta\,\Sigma\alpha\}$

$- 8\{\Sigma\alpha\,\Sigma\alpha\beta - 3\alpha\beta\gamma\}]$

$= -[-3p^3 - 12p^3 - 24pq + 8\{(-p)(-p)^2 - 2q(-p)\}$

$- 8\{(-p)q - 3(-r)\}]$

$= -[-15p^3 - \cancel{24pq} - 8p^3 + \cancel{16pq} + \cancel{8pq} - 24r]$

$= -[-23p^3 - 24r] = 23p^3 + 24r$ Ans.

Transformation of axes :—

Ex- 5D

1. (i) Given equ^n is $x^3 - 2x - \frac{3}{40} = 0$

multiplying the roots of the equ^n by 10. The transform equ^n is

$y^3 - 2 \cdot 10^2 \cdot y - 10^3 \cdot \frac{3}{40} = 0$

or, $y^3 - 200y - 75 = 0$.

Suitable constant is 10 Ans

Given equ^n is $x^3 + \frac{1}{2}x^2 + \frac{5}{36}x + \frac{7}{72} = 0$

(ii) multiplying the roots of the equ^n by 6. The transform equ^n is

$y^3 + \frac{1}{2}y^2 \cdot 6 + \frac{5}{36} \cdot 6^2 y + \frac{7}{72} \cdot 6^3 = 0$

or, $y^3 + 3y^2 + 5y + 216 = 0$

∴ Suitable constant is 6. Ans.

The given equ^n is $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ ①.

The roots of the equ^n are α, β, γ,

From the relation between roots & co-efficient we get,

$\cancel{\Sigma\alpha = -\frac{a_1}{a_0}}$, $\cancel{\Sigma\alpha\beta = \frac{a_2}{a_0}}$, $\alpha\beta\gamma = -\frac{a_3}{a_0}$.

(i) $\cancel{\Sigma\frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} =}$

Let $\alpha' = \frac{1}{\alpha}$, $\beta' = \frac{1}{\beta}$, $\gamma' = \frac{1}{\gamma}$;

Then the equ^n are $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$.

From the relation between roots and co-effint we get,

$\Sigma\alpha' = -\frac{a_2}{a_3}$ $\Sigma\alpha'\beta' = \frac{a_1}{a_3}$ $\alpha'\beta'\gamma' = -\frac{a_0}{a_3}$.

(i) $\Sigma\frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \Sigma\alpha'^2$

$= (\Sigma\alpha')^2 - 2\Sigma\alpha'\beta' = \frac{a_2^2}{a_3^2} - 2 \cdot \frac{a_1}{a_3} = \frac{a_2^2 - 2a_1a_3}{a_3^2}$. Ans.

(ii) $\Sigma\frac{1}{\alpha^2\beta^2} = (\Sigma\alpha'\beta')^2 - 2\Sigma\alpha'(\alpha'\beta'\gamma')$

$= \frac{a_1^2}{a_3^2} - 2\left(-\frac{a_2}{a_3}\right)\left(-\frac{a_0}{a_3}\right) = \frac{a_1^2 - 2a_2a_0}{a_3^2}$. Ans

(iii) $\Sigma\frac{1}{\alpha^3} = (\Sigma\alpha')^3 = \Sigma\alpha'^2 \cdot \Sigma\alpha' - \Sigma\alpha'^2\beta'$

$= \{(\Sigma\alpha')^2 - 2\Sigma\alpha'\beta'\}\Sigma\alpha' - \{\Sigma\alpha'\Sigma\alpha'\beta' - 3\alpha'\beta'\gamma'\}$

$= \left(\frac{a_2^2}{a_3^2} - 2\frac{a_1}{a_3}\right)\left(-\frac{a_2}{a_3}\right) - \left\{\left(-\frac{a_2}{a_3}\right)\left(\frac{a_1}{a_3}\right) - 3\left(-\frac{a_0}{a_3}\right)\right\}$

$= +\left(\frac{a_2^2 - 2a_1a_3}{a_3^2}\right)\left(-\frac{a_2}{a_3}\right) - \left\{\frac{3a_0}{a_3} + \frac{a_1a_2}{a_3^2}\right\}$

$= \frac{2a_1a_2a_3 - a_2^3 - 3a_0a_3^2 + a_1a_2a_3}{a_3^3}$

$= \frac{3a_1a_2a_3 - 3a_0a_3^2 - a_2^3}{a_3^3}$ Ans

6. The given equ^n is $x^3 + px^2 + qx + r = 0$ ① $\to \alpha, \beta, \gamma$.
from the relation between roots & co-efficient we get

$\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\alpha\beta\gamma = -r$,

(i) Now we shall find out an equ^n whose roots are,

$\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma}$, $\frac{1}{\alpha} + \frac{1}{\gamma} - \frac{1}{\beta}$, $\frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\alpha}$.

Let $y = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - \frac{2}{\gamma}$

or, $y = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma} - \frac{2}{\gamma}$

or, $y = \frac{q}{-r} - \frac{2}{\gamma}$

or, $\gamma = \frac{-2r}{q + ry}$

Since γ is a root of the equ^n ① we get,

$\gamma^3 + p\gamma^2 + q\gamma + r = 0$

or, $\left(\frac{-2r}{q+ry}\right)^3 + p\left(\frac{-2r}{q+ry}\right)^2 + q\left(\frac{-2r}{q+ry}\right) + r = 0$

or, $r(q+ry)^3 - 2rq(q+ry)^2 + 4r^2p(q+ry) - 8r^3 = 0$

(ii) Now we shall find out an equ^n whose roots are,

$\alpha\beta + \frac{1}{\gamma}$, $\beta\gamma + \frac{1}{\alpha}$, $\gamma\alpha + \frac{1}{\beta}$,

Let $y = \alpha\beta + \frac{1}{\gamma} = \frac{\alpha\beta\gamma + 1}{\gamma}$

or, $\gamma = \frac{-r+1}{y} = \frac{(1-r)}{y}$,

Since γ is a root of the equ^n ① we get,

$\left(\frac{1-r}{y}\right)^3 + p\left(\frac{1-r}{y}\right)^2 + q\left(\frac{1-r}{y}\right) + r = 0$

or, $ry^3 + q(1-r)y^2 + p(1-r)^2y + (1-r)^3 = 0$

(iii) Find out an equ^n whose roots are,

$\alpha - \frac{\beta\gamma}{\alpha}$, $\beta - \frac{\gamma\alpha}{\beta}$, $\gamma - \frac{\alpha\beta}{\gamma}$.

Let $y = \alpha - \frac{\beta\gamma}{\alpha} = \alpha + \beta + \gamma - \frac{\beta\gamma}{\alpha} - \beta - \gamma$

or, $y = \Sigma\alpha - \frac{\Sigma\alpha\beta}{\alpha} = (-p) - \frac{q}{\alpha}$

or, $\alpha = \frac{-q}{p+y}$

Since α is a root of the equ^n ① we get,

$\alpha^3 + q\alpha^2 + q\alpha + r = 0$

or, $\left(\frac{-q}{p+y}\right)^3 + q\left(\frac{-q}{p+y}\right)^2 + q\left(\frac{-q}{p+y}\right) + r = 0$

or, $r(p+y)^3 - q^2(p+y) + pq^2(p+y) - q^3 = 0$

(iv) Now we shall find out an equ^n whose roots are

$\frac{\alpha+\beta}{\gamma}$, $\frac{\beta+\gamma}{\alpha}$, $\frac{\gamma+\alpha}{\beta}$

Let $y = \frac{\alpha+\beta}{\gamma} = \frac{\alpha+\beta+\gamma}{\gamma} - 1 = \frac{\Sigma\alpha}{\gamma} - 1$

or, $y+1 = \frac{(-p)}{\gamma}$ or, $\gamma = \frac{-p}{y+1}$

since γ is the root of the equ^n we have,

$\left(\frac{-p}{y+1}\right)^3 + p\left(\frac{-p}{y+1}\right)^2 + q\left(\frac{-p}{y+1}\right) + r = 0$

or, $r(y+1)^3 - pq(y+1)^2 + p^3(y+1) - p^3 = 0$ Ans.

The equ^n is $x^3 + qx + r = 0$ $\Sigma\alpha = 0$, $\Sigma\alpha\beta = q$ $\alpha\beta\gamma = -r$

) $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$

Let $y = \alpha(\beta+\gamma) = \alpha(-\alpha) = -\alpha^2$ $[\because \alpha+\beta+\gamma = 0]$

$\therefore \alpha^2 = -y$

Since α is a root of the equ^n,

$x^3 + qx + r = 0$

or, $\alpha^3 + q\alpha + r = 0$

or, $\alpha^2(\alpha^2+q)^2 = r^2$

or, $(-y)(-y+q)^2 = r^2$

or, $y(y^2+q^2-2yq) = -r^2$

or, $y^3 - 2y^2q + q^2y + r^2 = 0$ Ans

) $(\alpha-\beta)(\alpha-\gamma)$, $(\beta-\alpha)(\beta-\gamma)$, $(\gamma-\alpha)(\gamma-\beta)$

Let $y = (\alpha-\beta)(\alpha-\gamma)$

or, $y = \alpha^2 - \alpha(\alpha+\beta) + \beta\gamma$

or, $y = \alpha^2 + \alpha^2 + \beta\gamma$

or, $y = 2\alpha^2 - \frac{r}{\alpha}$

or, $y\alpha = 2\alpha^3 - r$

or, $2\alpha^3 - y\alpha - r = 0$ (2)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

or, $q = \beta\gamma + \alpha(\beta+\gamma)$

or, $q = \beta\gamma + \alpha(-\alpha)$

or, $q = \beta\gamma - \alpha^2$

or, $\beta\gamma = q + \alpha^2$

$\alpha\beta\gamma = -r$

or, $\beta\gamma = -\frac{r}{\alpha}$

Since α is the root of the equ^n (1)

$$\begin{array}{r} 2\alpha^3 + 2q\alpha + 2r = 0 \\ 2\alpha^3 - y\alpha - r = 0 \\ - \quad + \quad + \\ \hline \alpha(2q+y) + 3r = 0 \end{array}$$

or, $\alpha = -\frac{3r}{2q+y}$

Putting this value of α in (1) we get,

$\left\{\frac{-3r}{(2q+y)}\right\}^3 + q\left\{\frac{-3r}{(2q+y)}\right\} + r = 0$

or, $r(2q+y)^3 - 3qr(2q+y) - 27r^3 = 0$ Ans

$\alpha^2+\beta^2$, $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$;

Let $y = \alpha^2+\beta^2 = \alpha^2+\beta^2+\gamma^2 - \gamma^2$

or, $y = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta - \gamma^2 = 0 - 2q - \gamma^2$

or, $\gamma^2 = -y - 2q$

Since γ is a root of the equ^n

$\gamma^3 + q\gamma + r = 0$

or, $\gamma(\gamma^2+q) = -r$

or, $\gamma^2(\gamma^2+q)^2 = r^2$

or, $(-y-2q)(-y-2q+q)^2 = r^2$

or, $(2q+y)(-y-q)^2 = -r^2$

or, $(2q+y)(y^2+2qy+q^2) = r^2$

Let $y = \frac{\alpha+\beta}{\gamma} = \frac{\alpha+\beta+\gamma}{\gamma} - 1 = \frac{\Sigma\alpha}{\gamma} - 1$

or, $y+1 = \frac{(-p)}{\gamma}$ or, $\gamma = \frac{-p}{y+1}$

Since γ is the root of the equⁿ we have,

$\left(\frac{-p}{y+1}\right)^3 + p\left(\frac{-p}{y+1}\right)^2 + q\left(\frac{-p}{y+1}\right) + r = 0$

or, $r(y+1)^3 - pq(y+1)^2 + p^3(y+1) - p^3 = 0$ Ans.

The equⁿ is $x^3 + qx + r = 0$ $\Sigma\alpha = 0$, $\Sigma\alpha\beta = q$ $\alpha\beta\gamma = -r$

) $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$

Let $y = \alpha(\beta+\gamma) = \alpha(-\alpha) = -\alpha^2$ $[\because \alpha+\beta+\gamma = 0]$

$\therefore \alpha^2 = -y$

Since α is a root of the equⁿ,

$x^3 + qx + r = 0$

or, $\alpha^3 + q\alpha + r = 0$

or, $\alpha^2(\alpha^2+q)^2 = r^2$.

or, $(-y)(-y+q)^2 = r^2$

or, $y(y^2+q^2-2yq) = -r^2$

or, $y^3 - 2y^2q + q^2y + r^2 = 0$ Ans

) $(\alpha-\beta)(\alpha-\gamma)$, $(\beta-\alpha)\cdot(\beta-\gamma)$, $(\gamma-\alpha)(\gamma-\beta)$

Let $y = (\alpha-\beta)(\alpha-\gamma)$

or, $y = \alpha^2 - \alpha(\alpha+\beta) + \beta\gamma$

or, $y = \alpha^2 + \alpha^2 + \beta\gamma$

or, $y = 2\alpha^2 - \frac{r}{\alpha}$

or, $y\alpha = 2\alpha^3 - r$

or, $2\alpha^3 - y\alpha - r = 0$ (2)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

or, $q = \beta\gamma + \alpha(\beta+\gamma)$

or, $q = \beta\gamma + \alpha(-\alpha)$

or, $q = \beta\gamma - \alpha^2$

or, $\beta\gamma = q + \alpha^2$

$\alpha\beta\gamma = -r$

or, $\beta\gamma = -\frac{r}{\alpha}$

Since α is the root of the equⁿ (1)

$$\begin{array}{r} 2\alpha^3 + 2q\alpha + 2r = 0 \\ -\cancel{2\alpha^3} \overset{+}{-} y\alpha \overset{+}{-} r = 0 \\ \hline \alpha(2q+y) + 3r = 0 \end{array}$$

or, $\alpha = -\frac{3r}{2q+y}$

Putting this value of α in (1) we get,

$\left\{\frac{-3r}{(2q+y)}\right\}^3 + q\left\{\frac{-3r}{(2q+y)}\right\} + r = 0$

or, $r(2q+y)^3 - 3qr(2q+y) - 27r^3 = 0$ Ans

) $\alpha^2+\beta^2$, $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$;

Let $y = \alpha^2+\beta^2 = \alpha^2+\beta^2+\gamma^2 - \gamma^2$

or, $y = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta - \gamma^2 = 0 - 2q - y^2$

or, $\gamma^2 = -y - 2q$

Since γ is a root of the equⁿ

$\gamma^3 + q\gamma + r = 0$

or, $\gamma(\gamma^2+q) = -r$

or, $\gamma^2(\gamma^2+q)^2 = r^2$

or, $(-y-2q)(-y-2q+q)^2 = r^2$

or, $(2q+y)(-y-q)^2 = -r^2$

or, $(2q+y)(y^2+2qy+q^2) = r^2$

Given equ^n is $x^3+6x^2+12x+35=0$ ①.

1. Let us apply transformation $x=y+h$ so that the transform equ^n may want the 2nd term.

The transform equ^n is

$(y+h)^3+6(y+h)^2+12(y+h)+35=0$

$y^3+(3h+6)h^2+(3h^2+12h+12)y+(h^3+6h^2+12h+35)=0$,

or, By the given con^n $3h+6=0$ or, $h=-2$,

The equ^n reduces to $y^3+27=0$

Solving equ^n ② we get $y=-3,-3\omega,-3\omega^2$,

where ω is the imaginary cube root of unity,

Since $x=y-2$.

∴ Roots of the given equ^n are $x=-3-2=-5$,

$x=-3\omega-2$, $x=-3\omega^2-2$. Ans.

Given equ^n is $x^4+4x^3+9x^2+10x-6=0$ ①

Let us apply transformation $x=y+h$ so that the transform equ^n may want the 2nd term.

The transform equ^n is

$(y+h)^4+4(y+h)^3+9(y+h)^2+10(y+h)-6=0$

or, $(y^4+4y^3h+6y^2h^2+4yh^3+h^4)+4(y^3+3y^2h+3yh^2+h^3)+10y+10h-6=0$

$+9(y^2+h^2+2yh)$

or, $y^4+(4h+4)y^3+(6h^2+12h+9)y^2+(4h^3+12h^2+18h+10)y+(h^4+4h^3+9h^2+10h-6)=0$

By the given con^n, $4h+4=0$

or, $h=-1$,

The equ^n reduces to,

$y^4+(6-12+9)y^2+(-4+12-18+10)y+(1-4+9-10-6)=0$

or, $y^4+3y^2-10=0$ Ans.

6. The given equ^n is $a_0x^3+3a_1x^2+3a_2x+a_3=0$

Let us apply transformation $x=y+h$ so that the transform may want the 2nd term.

The transform equ^n is

$a_0(y+h)^3+3a_1(y+h)^2+3a_2(y+h)+a_3=0$

or, $a_0\{y^3+h^3+3y^2h+3yh^2\}+3a_1\{y^2+h^2+2yh\}+3a_2y+3a_2h+a_3=0$

or, $a_0y^3+a_0h^3+3a_0y^2h+3a_0yh^2+3a_1y^2+3a_1h^2+6a_1yh+3a_2y+3a_2h+a_3=0$

or, $a_0y^3+(3a_0h+3a_1)y^2+(3a_0h^2+6a_1h+3a_2)y+(a_0h^3+3a_1h^2+3a_2h+a_3)=0$

By the given con^n, $3a_0h+3a_1=0$

or, $h=-\frac{a_1}{a_0}$,

~~Then the equ^n reduces to,~~

$$3a_0h^2 + 6a_1h + 3a_2 = 0$$

or, $3a_0\left(-\frac{a_1}{a_0}\right)^2 + 6a_1\left(-\frac{a_1}{a_0}\right) + 3a_2 = 0$

or, $3a_1^2 - 6a_1^2 + 3a_2a_0 = 0$

or, $a_1^2 = a_0a_2$

which is the required relation. $\underline{Ans}$

17. The given equ^n is $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, (1)
Let us apply transformation $x = y + h$ so that the transform equ^n may removed 2nd term and 4th term.
The transform equ^n is
$a_0(y+h)^4 + 4a_1(y+h)^3 + 6a_2(y+h)^2 + 4a_3(y+h) + a_4 = 0$
or, $a_0(y^4 + 4y^3h + 6y^2h^2 + 4yh^3 + h^4) + 4a_1(y^3 + h^3 + 3y^2h + 3yh^2)$
$+ 6a_2(y^2 + h^2 + 2yh) + 4a_3y + 4a_3h + a_4 = 0$
or, $a_0y^4 + (4a_0h + 4a_1)y^3 + (6a_0h^2 + 12a_1h^2 + 6a_2)y^2$
$+ (4a_0h^3 + 12a_1h^2 + 12a_2h + 4a_3)y + (a_0h^4 + 4a_1h^3 + 6a_2h^2 + 4a_3h + h^4) = 0$
By the given con^n, we have,
$4a_0h + 4a_1 = 0$ or, $h = -\frac{a_1}{a_0}$

$4a_0h^3 + 12a_1h^2 + 12a_2h + 4a_3 = 0$

or, $4a_0\left(-\frac{a_1}{a_0}\right)^3 + 12a_1\left(-\frac{a_1}{a_0}\right)^2 + 12a_2\left(-\frac{a_1}{a_0}\right) + 4a_3 = 0$

or, $-4a_1^3 + 12a_1^3 - 12a_0a_1a_2 + 4a_0^2a_3 = 0$

or, $2a_1^3 - 3a_0a_1a_2 + a_0^2a_3 = 0$

which is the required relation. $\underline{A}$

18. The given equ^n is $4x^3 - 8x^2 - 19x + 26 = 0$ (1).
Let α, β, γ are the roots of the equ^n (1)
To find the equ^n whose roots are
$\alpha - 2, \beta - 2, \gamma - 2$.
Let $y = \alpha - 2$ or, $\alpha = y + 2$.
Then the equ^n becomes,
$4(y+2)^3 - 8(y+2)^2 - 19(y+2) + 26 = 0$
or, $4(y^3 + 6y^2 + 12y + 8) - 8(y^2 + 4 + 4y) - 19y - 38 + 26 = 0$
or, $4y^3 + (24-8)y^2 + (48 - 32 - 19)y + 32 - 38 + 26 = 0$
-32
or, $4y^3 + 16y^2 - 3y - 12 = 0$

$f(x) = 4x^3 - 8x^2 - 19x + 26$.
$f(-x) = -4x^3 - 8x^2 + 19x + 26$.
$\phi(y) = 4y^3 + 16y^2 - 3y - 12$
$\phi(-y) = -4y^3 + 16y^2 + 3y - 12$

The sign in the sequence of the co-effitient of $f(x)$ are
$+ \; - \; - \; +$. There are $f(x)$ has atmost 2 +ve roots.

The sign in the sequen -- $f(-x)$ are $- \; - \; + \; +$.
There are $f(-x)$ has exactly one -ve root.

$\phi(y)$ The sign in the sequence of the co-effitient of $\phi(y)$ are
$+ \; + \; - \; -$ $\phi(y)$ has only one +ve root
The sign in the sequence of the co-effitient of $\phi(-y)$ are
$- \; + \; + \; -$.
There are two chang of sign.
$\phi(-y)$ has atmost two -ve root,
Since roots of the equ^n $\phi(y)=0$ is diminished by 2 of the root of the $f(x)=0$, if $f(x)=0$ has no +ve real root then $\phi(y)=0$ has no +ve real root.
which is a contradiction,
$\therefore f(x)=0$ has exactly two +ve and one -ve real root.

19. Given equ^n is $x^4+3x^2+8x+3=0$ (1)
$f(x) = x^4+3x^2+8x+3$.
Let $\alpha, \beta, \gamma, \delta$ are the roots of the equ^n (1).
To find the equ^n whose roots are,
$\alpha+1, \beta+1, \gamma+1, \delta+1$.
Let $y=\alpha+1$ or, $\alpha=y-1$.
Then the equ^n becomes,
$(y-1)^4+3(y-1)^2+8(y-1)+3=0$
or, $(y^4-4y^3+6y^2-4y+1)+3(y^2-2y+1)+8y-8+3=0$
or, $y^4-4y^3+9y^2-2y-1=0$.
$\phi(y) = y^4-4y^3+9y^2-2y-1$.

$f(x) = + \; + \; + \; +$ $\quad f(-x) = + \; - \; - \; +$
$\phi(y) = + \; - \; + \; - \; -$ $\quad \phi(-y) = + \; + \; + \; + \; -$

	+ve	-ve	imaginary
$f(x)$	0	2/0	2
$\phi(y)$	3/1	1	2

If $f(x)=0$ has, no -ve real root then all the roots of $f(x)=0$ are imaginary.
Since roots of the equ^n $\phi(y)=0$ are exceed the root of the equ^n $f(x)=0$ by $\cancel{0}$ 1. Then all the roots of $\phi(y)=0$ has exactly one -ve real root.
$\therefore$ No of real root of $f(x)=0$ is 2 and no of imaginary root of $f(x)=0$ is 2. Ans.

20. The given equⁿ is $x^4 - x^3 + 2x^2 - x + 1 = 0$ (1)

Let $f(x) = x^4 - x^3 + 2x^2 - x + 1$.

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equⁿ (1).

To find the equⁿ whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$.

Let $y = \alpha^2$,

Since α is a root of the given equⁿ,

$$\alpha^4 - \alpha^3 + 2\alpha^2 - \alpha + 1 = 0$$

or, $(\alpha^4 + 2\alpha^2 + 1) = \alpha(\alpha^2 + 1)$

or, $(\alpha^4 + 2\alpha^2 + 1)^2 = \alpha^2(\alpha^2 + 1)^2$

or, $(y^2 + 2y + 1)^2 = y(y + 1)^2$

or, $\{(y^2 + 1) + 2y\}^2 = y(y^2 + 1 + 2y)$

or, $y^4 + 3y^3 + 4y^2 + 3y + 1 = 0$ say $\phi(y) = 0$,

$\phi(y) = y^4 + 3y^3 + 4y^2 + 3y + 1$,

$+ \quad + \quad + \quad + \quad +$

Number of +ve real root is 0,

$\phi(-y) = y^4 - 3y^3 + 4y^2 - 3y + 1$,

$+ \quad - \quad + \quad - \quad +$

Number of −ve real root is 4 or 2 or 0.

Since $\phi(y) = 0$ has no +ve real root and roots of the equⁿ $\phi(y) = 0$ are square of the roots of the equⁿ $f(x) = 0$,

$\therefore f(x) = 0$ has no real root. [Proved]

21. The given equⁿ is $x^3 - ax^2 + bx - 1 = 0$

Let $f(x) = x^3 - ax^2 + bx - 1$

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equⁿ,

To find the equⁿ whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$

Let $y = \alpha^2$

Since α is a root of the given equⁿ,

$$\alpha^3 - a\alpha^2 + b\alpha - 1 = 0$$

or, $\alpha^3 + b\alpha = a\alpha^2 + 1$

or, $\alpha(\alpha^2 + b) = a\alpha^2 + 1$

or, $\alpha^2(\alpha^2 + b)^2 = (a\alpha^2 + 1)^2$

or, $\alpha^2(\alpha^4 + b^2 + 2b\alpha^2) = a^2\alpha^4 + 1 + 2\alpha^2 a$

or, $y(y^2 + b^2 + 2by) = a^2y^2 + 1 + 2ay$

or, $y^3 + (2b - a^2)y^2 + (b^2 - 2a)y - 1 = 0$ (2)

By the given conⁿ equⁿ (2) is identical with (1).

we get, $2b - a^2 = -a$ & $b^2 - 2a = b$

$\therefore$ the condⁿ are $2p^3 - 9pq + 27r = 0$ Ans

26. Given equⁿ is $x^3 - 3x - 1 = 0$ ①
Since α is the root of the equⁿ ① then,
$$\alpha^3 - 3\alpha - 1 = 0 \quad ②$$
Let $y = 2 - \alpha^2$, or, $\alpha^2 = (2 - y)$,
From ② we get, $\alpha^3 - 3\alpha - 1 = 0$
or, $\alpha(\alpha^2 - 3) = 1$ or, $\alpha^2(\alpha^2 - 3)^2 = 1$
or, $(2 - y)\{2 - y - 3\}^2 = 1$
or, $(2 - y)(y^2 + 1 + 2y) = 1$
or, $y^3 - 3y + 1 = 0$ ③
Here the equⁿ ③ is same as equⁿ ①
$\therefore$ $(2 - \alpha^2)$ is also the root of the equⁿ ①.
Let other root ① is β.
$$\alpha + 2 - \alpha^2 + \beta = 0$$
or, $\beta = (\alpha^2 - \alpha - 2)$
$\therefore$ other roots of the given equⁿ are $(2 - \alpha^2)$ & $(\alpha^2 - \alpha - 2)$. Ans

27. Given equⁿ is $x^3 + 3x^2 - 6x + 1 = 0$ ①
Since α is a root of ① we have,
$$\alpha^3 + 3\alpha^2 - 6\alpha + 1 = 0 \quad ②$$
Let $y = \frac{1}{1 - \alpha}$ or, $\alpha = \frac{y - 1}{y}$,
From ② we have,
$$\left(\frac{y-1}{y}\right)^3 + 3\left(\frac{y-1}{y}\right)^2 - 6\left(\frac{y-1}{y}\right) + 1 = 0$$
or, $y^3 + 3y^2 - 6y + 1 = 0$ ③
Here the equⁿ ③ is same as equⁿ ①,
$\therefore$ $\frac{1}{1-\alpha}$ is also a root of ①.
Let β is also a root of ①,
Then, $\alpha + \frac{1}{1-\alpha} + \beta = 0$
or, $\beta = \frac{\alpha - 1}{\alpha}$. Ans

$x^3 - 3px^2 + 3(p-1)x + 1 = 0$ ① $\rightarrow \alpha, \beta, \gamma$.
We shall find out an equⁿ whose roots are $1-\alpha, 1-\beta, 1-\gamma$.
Let $y = 1 - \alpha$ or, $\alpha = (1 - y)$.
$$\alpha^3 - 3p\alpha^2 + 3(p-1)\alpha + 1 = 0$$
or, $(1-y)^3 - 3p(1-y)^2 + 3(p-1)(1-y) + 1 = 0$
or, $y^3 + 3(p-1)y^2 - 3py + 1 = 0$ ②.
From ① & ② we see that roots of the equⁿ ② are reciprocal of the root of the equⁿ ①.
$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are also the roots of the equⁿ ①

Again, $1-\alpha, 1-\beta, 1-\gamma$ are the roots of the equⁿ (2). (iii)

Now, $\alpha+\beta+\gamma = 3p$ (3)

If possible, let α, β be complex, γ is real,

p is real, then from (3) we have

$\frac{1}{\gamma} = 1-\gamma$

$\gamma^2 - \gamma + 1 = 0$,

or, $\omega = \frac{-1 \pm i\sqrt{3}}{2}$,

$\gamma = \omega$ and ω^2 or,

which is a complex → a contradiction,

∴ All the roots of the equⁿ are all real. Ans

Ex — 5E Reciprocal Equⁿ

1. (i) The given equⁿ is $x^4 + x^3 + 2x^2 + x + 1 = 0$,

This is a reciprocal equⁿ of 1st type,

This can be written as $(x^4+1) + (x^3+x) + 2x^2 = 0$

or, $\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 2 = 0$ [Dividing both side by x^2]

or, $\left\{\left(x+\frac{1}{x}\right)^2 - 2x\cdot\frac{1}{x}\right\} + \left(x + \frac{1}{x}\right) + 2 = 0$

or, $t^2 + t = 0$ [Puting $t = x + \frac{1}{x}$]

$t = 0, -1$,

when $t = 0$,

$x + \frac{1}{x} = 0$

or, $x = \pm i$

when $t = -1$

$x^2 + x + 1 = 0$

$x = \omega, \omega^2$

$= \frac{-1 \pm i\sqrt{3}}{2}$,

∴ The roots of the equⁿ are $\pm i$, $\frac{-1 \pm i\sqrt{3}}{2}$ Ans,

(ii) The given equⁿ is $x^4 - 8x^3 + 17x^2 - 8x + 1 = 0$.

This is a reciprocal equⁿ of 1st type,

This can be written as $(x^4+1) - 8(x^3+x) + 17x^2 = 0$

or, $\left(x^2 + \frac{1}{x^2}\right) - 8\left(x + \frac{1}{x}\right) + 17 = 0$ [Divide both side by x^2]

or, $\left\{\left(x+\frac{1}{x}\right)^2 - 2x\cdot\frac{1}{x}\right\} - 8\left(x + \frac{1}{x}\right) + 17 = 0$

or, $\left(x+\frac{1}{x}\right)^2 - 8\left(x+\frac{1}{x}\right) + 15 = 0$ [Puting $x + \frac{1}{x} = t$]

or, $t^2 - 8t + 15 = 0$

or, $(t-3)(t-5) = 0$

$t = 3, 5$,

when $t = 3$

$x + \frac{1}{x} = 3$

or, $x^2 - 3x + 1 = 0$

$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

when $t = 5$

$x + \frac{1}{x} = 5$

or, $x^2 - 5x + 1 = 0$

$x = \frac{5 \pm \sqrt{21}}{2}$

∴ The roots of the equⁿ are $\frac{3 \pm \sqrt{5}}{2}$, $\frac{5 \pm \sqrt{21}}{2}$,

(iii) The given equ^n is $x^4 - 4x^3 + 3x^2 - 4x + 1 = 0$

This is a reciprocal equ^n of 1st type.

This can be written as, $(x^4+1) - 4(x^3+x) + 3x^2 = 0$ [Dividing x^2 by both side]

or, $\left(x^2+\frac{1}{x^2}\right) - 4\left(x+\frac{1}{x}\right) + 3 = 0$

or, $\left\{\left(x+\frac{1}{x}\right)^2 - 2\cdot x\cdot\frac{1}{x}\right\} - 4\left(x+\frac{1}{x}\right) + 3 = 0$

or, $\left(x+\frac{1}{x}\right)^2 - 4\left(x+\frac{1}{x}\right) + 1 = 0$

or, $t^2 - 4t + 1 = 0$ [Putting $t = x + \frac{1}{x}$]

$t = \frac{4 \pm \sqrt{16-4}}{2\cdot 1} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$,

when $t = 2+\sqrt{3}$,

or, $x + \frac{1}{x} = 2+\sqrt{3}$

or, $x^2 + 1 = (2+\sqrt{3})x$

or, $x^2 - (2+\sqrt{3})x + 1 = 0$

$x = \frac{(2+\sqrt{3}) \pm \sqrt{(2+\sqrt{3})^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$

$= \frac{(2+\sqrt{3}) \pm \sqrt{\cancel{4}+3+4\sqrt{3}-\cancel{4}}}{2}$

$= \frac{(2+\sqrt{3}) \pm \sqrt{3+4\sqrt{3}}}{2}$

$t = 2-\sqrt{3}$

or, $x + \frac{1}{x} = 2-\sqrt{3}$

or, $x^2 - (2-\sqrt{3})x + 1 = 0$

$x = \frac{(2-\sqrt{3}) \pm \sqrt{(2-\sqrt{3})^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$

$= \frac{(2-\sqrt{3}) \pm \sqrt{4+3-4\sqrt{3}-4}}{2}$

$= \frac{(2-\sqrt{3}) \pm \sqrt{3-4\sqrt{3}}}{2}$,

$\therefore$ The roots of the equ^n are $\frac{(2+\sqrt{3}) \pm \sqrt{3+4\sqrt{3}}}{2}$ and $\frac{(2-\sqrt{3}) \pm \sqrt{3-4\sqrt{3}}}{2}$ A

) The given equ^n is $2x^5 + 5x^4 - 5x - 2 = 0$,

This is a reciprocal equ^n of 2nd type,

This can be written as $(2x^5 - 2) + 5(x^4 - x) = 0$

or, $2(x-1)(x^4 + x^3 + x^2 + x + 1) + 5x(x-1)(x^2+x+1) = 0$

or, $(x-1)\{2x^4 + 2x^3 + 2x^2 + 2x + 1 + 5x^3 + 5x^2 + 5x\} = 0$

$x = 1$ | $2x^4 + 7x^3 + 7x^2 + 7x + 2 = 0$

$\therefore$ $2x^4 + 7x^3 + 7x^2 + 7x + 2 = 0$ is a reciprocal equ^n of even degree and of 1st type.

$\therefore$ This is of the standard form.

This can be written as $(2x^4 + 2) + (7x^3 + 7x) + 7x^2 = 0$

or, $2\left\{\left(x+\frac{1}{x}\right)^2 - 2\cdot x\cdot\frac{1}{x}\right\} + 7\left(x+\frac{1}{x}\right) + 7 = 0$

or, $2\left(x+\frac{1}{x}\right)^2 + 7\left(x+\frac{1}{x}\right) + 3 = 0$

or, $2t^2 + 7t + 3 = 0$ [Putting $x + \frac{1}{x} = t$]

or, $(t+3)(2t+1) = 0$

$t = -3$, $t = -\frac{1}{2}$

when $t = -3$

or, $x + \frac{1}{x} = -3$

or, $x^2 + 3x + 1 = 0$

$x = \frac{-3 \pm \sqrt{9 - 4.1.1}}{2.1}$

$= \frac{-3 \pm \sqrt{5}}{2}$

when $t = -\frac{1}{2}$,

$x + \frac{1}{x} = -\frac{1}{2}$

or $2x^2 + x + 2 = 0$

$x = \frac{-1 \pm \sqrt{1 - 4.2.2}}{2.2}$

$= \frac{-1 \pm \sqrt{15}\, i}{4}$

$\therefore$ The roots of the equⁿ are

$\frac{-3 \pm \sqrt{5}}{2}$ and $\frac{-1 \pm \sqrt{15}\, i}{4}$ and 1 Ans.

(v) Given equⁿ is $3x^5 + 7x^4 - 4x^3 + 4x^2 - 7x - 3 = 0$

This is a reciprocal equⁿ of 2nd degree, type.

This can be written as.

$(3x^5 - 3) + (7x^4 - 7x) - (4x^3 - 4x^2) = 0$

or, $3(x^5 - 1) + 7x(x^3 - 1) - 4x^2(x - 1) = 0$

or, $3(x-1)(x^4 + x^3 + x^2 + x + 1) + 7x(x-1)(x^2 + x + 1) - 4x^2(x-1) = 0$

or, $(x-1)\{3x^4 + 3x^3 + 3x^2 + 3x + 3 + 7x^3 + 7x^2 + 7x - 4x^2\} = 0$

or, $(x-1)\{3x^4 + 10x^3 + 6x^2 + 10x + 3\} = 0$

$x = 1$ | $3x^4 + 10x^3 + 6x^2 + 10x + 3 = 0$

$\therefore$ $3x^4 + 10x^3 + 6x^2 + 10x + 3 = 0$ is a reciprocal equⁿ of even degree and of 1st type.

This is of the ~~2nd term~~ standard form,

This can be written as,

$(3x^4 + 3) + (10x^3 + 10x) + 6x^2 = 0$

or, $3\left(x^2 + \frac{1}{x^2}\right) + 10\left(x + \frac{1}{x}\right) + 6 = 0$ [Dividing x^2 both side)

or, $3\left\{\left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x}\right\} + 10\left(x + \frac{1}{x}\right) + 6 = 0$

or, $3\left(x + \frac{1}{x}\right)^2 + 10\left(x + \frac{1}{x}\right) = 0$

or, $3t^2 + 10t = 0$ (Puti $t = x + \frac{1}{x}$)

or, $t = 0$

$3t + 10 = 0$

or, $t = -\frac{10}{3}$,

when $t = 0$

$x + \frac{1}{x} = 0$

or $x^2 + 1 = 0$

$x = \pm i$

$x + \frac{1}{x} = -\frac{10}{3}$

or, $3(x^2 + 1) + 10x = 0$

or, $3x^2 + 10x + 3 = 0$

or, $(x + 3)(3x + 1) = 0$

$x = -3, -\frac{1}{3}$,

$\therefore$ The roots of the equⁿ are $\pm i, -3, -\frac{1}{3}, 1$ Ans

(i) Given equ^n is $2x^5 - 3x^4 - x^3 - x^2 - 3x + 2 = 0$

This is a reciprocal equ^n of 1st type.

This can be written as,

$(2x^5+2) - (3x^4+3x) - (x^3+x^2) = 0$

or, $2(x^5+1) - 3x(x^3+1) - x^2(x+1) = 0$

or, $2(x+1)(x^4-x^3+x^2-x+1) - 3x(x+1)(x^2-x+1) - x^2(x+1) = 0$

or, $(x+1)\{2x^4 - 5x^3 + 4x^2 - 5x + 2\} = 0$

$x+1=0$ | $2x^4 - 5x^3 + 4x^2 - 5x + 2 = 0$

or, $x=-1$ | This is a reciprocal equ^n of even degree and of 1st type.

This is of the standard form.

This can be written as,

$2(x^4+1) - 5(x^3+x) + 4x^2 = 0$ [Dividing both side by x^2]

or, $2\{x^2 + \frac{1}{x^2}\} - 5(x+\frac{1}{x}) + 4 = 0$

or, $2\{(x+\frac{1}{x})^2 - 2x\cdot\frac{1}{x}\} - 5(x+\frac{1}{x}) + 4 = 0$ [Putting $x+\frac{1}{x} = t$]

or, $2t^2 - 5t = 0$

$t = 0$, $t = \frac{5}{2}$, $x + \frac{1}{x} = \frac{5}{2}$

when $t = 0$

$x + \frac{1}{x} = 0$

or, $x = \pm i$

or, $2x^2 - 5x + 2 = 0$

$x = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5\pm 3}{4}$

$x = \frac{5+3}{4} = 2$, $x = \frac{5-3}{4} = \frac{1}{2}$.

∴ The root of the equ^n are $-1, \pm i, 2, \frac{1}{2}$. A

(ii) The given equ^n is $x^6 - 8x^4 + 8x^2 - 1 = 0$,

This is a reciprocal equ^n of 2nd type,

This can be written as,

$(x^6-1) - 8(x^4-x^2) = 0$

or, $(x-1)(x^5+x^4+x^3+x^2+x+1) - 8x^2(x^2+1)(x-1) = 0$

or, $(x-1)\{x^5+x^4+x^3+x^2+x+1 - 8x^3 - 8x^2\} = 0$

or, $(x-1)\{x^5+x^4-7x^3-7x^2+x+1\} = 0$

$x=1$ | $x^5+x^4-7x^3-7x^2+x+1=0$,

| This is a reciprocal equ^n of odd degree and of 1st type.

This can be written as,

$(x^5+1) + (x^4+x) - 7(x^3+x^2) = 0$

or, $(x+1)(x^4-x^3+x^2-x+1) + x(x+1)(x^2-x+1) - 7x^2(x+1) = 0$

or, $(x+1)\{x^4-x^3+x^2-x+1 + x^3-x^2+x - 7x^2\} = 0$

$x+1=0$ | $x^4 - 7x^2 + 1 = 0$

$x=-1$ | This is a reciprocal equ^n of 1st type and even deg.

This is of the standard form,

This can be written as,

$(x^4+1) - 7x^2 = 0$ [Dividing x^2 both side]

or, $(x^2 + \frac{1}{x^2}) - 7 = 0$

or, $t(t^2 - t - 2) = 0$

$t = 0$ $\qquad t^2 - t - 2 = 0$ $\qquad$ or, $(t-2)(t+1) = 0$

$t = 2, -1,$

when $t = 0$ $\qquad$ when $t = 2$ $\qquad$ when $t = -1$

$x + \frac{1}{x} = 0$ $\qquad x + \frac{1}{x} = 2$ $\qquad$ or, $x + \frac{1}{x} = -1$

or, $x = \pm i$ $\qquad$ or $x^2 - 2x + 1 = 0$ $\qquad$ or, $x^2 + x + 1 = 0$

$x = \frac{2 \pm \sqrt{4 - 4.1.1}}{2}$ $\qquad x = \frac{-1 \pm \sqrt{3}i}{2}$,

$= 1, 1,$

∴ the roots of the eqn are $1, 1, \pm i, \frac{-1 \pm \sqrt{3}i}{2}$ Ans

Given eqn is $x^7 + 4x^6 + 4x^5 + x^4 - x^3 - 4x^2 - 4x - 1 = 0$,

This is a reciprocal eqn of 2nd type,

this can be written as, $(x^7 - 1) + (4x^6 - 4x) + (4x^5 - 4x^2) + (x^4 - x^3) = 0$

or, $(x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) + 4x(x-1)(x^4 + x^3 + x^2 + x + 1)$

$+ 4x^2(x-1)(x^2 + x + 1) + x^3(x-1) = 0$

or, $(x-1)\{x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 + 4x^5 + 4x^4 + 4x^3 + 4x^2 + 4x$

$+ 4x^4 + 4x^3 + 4x^2 + x^3\} = 0$

or, $(x-1) = 0$ $\quad$ $x^6 + 5x^5 + 9x^4 + 10x^3 + 9x^2 + 5x + 1\} = 0$

$x = 1.$ $\quad$ this is a reciprocal eqn of even degree and of 1st type,

this can be written as,

$(x^6 + 1) + 5(x^5 + x) + 9(x^4 + x^2) + 10x^3 = 0$

$(x^3 + \frac{1}{x^3}) + 5(x^2 + \frac{1}{x^2}) + 9(x + \frac{1}{x}) + 10 = 0$

or, $(x + \frac{1}{x})\{(x^2 + \frac{1}{x^2} - 1\} + 5\{(x + \frac{1}{x})^2 - 2.x.\frac{1}{x}\} + 9(x + \frac{1}{x}) + 9 = 0$

or, $(x + \frac{1}{x})\{(x + \frac{1}{x})^2 - 2x.\frac{1}{x} - 1\} + 5\{(x + \frac{1}{x})^2 - 2\} + 9(x + \frac{1}{x}) + 9 = 0$

or, $(x + \frac{1}{x})\{(x + \frac{1}{x})^2 - 3\} + 5(x + \frac{1}{x})^2 - 10 + 9(x + \frac{1}{x}) + 9 = 0$

or, $t(t^2 - 3) + 5t^2 + 9t - 1 = 0$ (Putting $x + \frac{1}{x} = t$)

or, $t^3 + 5t^2 + 6t = 0$

or, $t(t^2 + 5t + 6) = 0$ $\qquad x + \frac{1}{x} = -2$

$t = 0,$ $\qquad t^2 + 5t + 6 = 0$ $\qquad$ or, $x^2 + 2x + 1 = 0$

$x + \frac{1}{x} = 0$ $\qquad \cancel{x = -5 \pm \sqrt{}}$ $\qquad x = \frac{-2 \pm \sqrt{4 - 4.1.1}}{2.1}$

or, $t = -2, t = -3$

or, $x = \pm i,$ $\qquad = -\frac{2}{2} = -1$

$x + \frac{1}{x} = -3$ $\qquad x = -1, -1$

or, $x^2 + 3x + 1 = 0$

$x = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2},$

5. Given equⁿ is $x^4 - 8x^3 + 20x^2 - 24x + 12 = 0$ ⓵

Let $\alpha, \beta, \gamma, \delta$ are the four roots of the equⁿ ⓵.

We shall find out an equⁿ whose roots are $(\alpha-1), (\beta-1), (\gamma-1), (\delta-1)$.

Let $y = \alpha - 1$

or, $\alpha = y + 1$.

Since α is a root of the equⁿ ⓵,

$\alpha^4 - 8\alpha^3 + 20\alpha^2 - 24\alpha + 12 = 0$

or, $(y+1)^4 - 8(y+1)^3 + 20(y+1)^2 - 24(y+1) + 12 = 0$

or, $(y^4 + 4y^3 + 6y^2 + 4y + 1) - 8(y^3 + 1 + 3y^2 + 3y) + 20(y^2 + 1 + 2y)$
$- 24y - 24 + 12 = 0$

or, $y^4 + (4-8)y^3 + (6 - 24 + 20)y^2 + (4 - 24 + 40 - 24)y$
$+ 1 - 8 + 20 - 24 + 12 = 0$

or, $y^4 - 4y^3 + 2y^2 - 4y + 1 = 0$ ⓶

This is a reciprocal equⁿ of even degree and 1st type. The equⁿ is of a standard form.

The equⁿ ⓶ can be written as,

$(y^4 + 1) - 4(y^3 + y) + 2y^2 = 0$

or, $\left(y^2 + \frac{1}{y^2}\right) - 4\left(y + \frac{1}{y}\right) + 2 = 0$ [Dividing y^2 by]

or, $\left\{\left(y + \frac{1}{y}\right)^2 - 2 \cdot y \cdot \frac{1}{y}\right\} - 4\left(y + \frac{1}{y}\right) + 2 = 0$ $\left[y + \frac{1}{y} = t\right]$

or, $t^2 - 4t = 0$

$t = 0$, $t = 4$

$y + \frac{1}{y} = 0$ $y + \frac{1}{y} = 4$

or, $y = \pm i$ or, $y^2 - 4y + 1 = 0$

$y = (2 \pm \sqrt{3})$,

Since $x = y + 1$

∴ The roots of the given equⁿ are $(1 \pm i), (3 \pm \sqrt{3})$ Ans.

6. Given equⁿ is $x^4 + 7x^3 + 20x^2 + 27x + 15 = 0$ ⓵

Let $\alpha, \beta, \gamma, \delta$ are the four roots of the equⁿ ⓵.

We shall find out an equⁿ whose roots are $(\alpha+2), (\beta+2), (\gamma+2), (\delta+2)$.

Let $y = \alpha + 2$

or, $\alpha = (y - 2)$,

Since α is a root of the equⁿ ⓵,

$\alpha^4 + 7\alpha^3 + 20\alpha^2 + 27\alpha + 15 = 0$

or, $(y-2)^4+7(y-2)^3+20(y-2)^2+27(y-2)+15=0$,

or, $y^4-4y^3\cdot 2+6y^2\cdot 2^2-4y\cdot 2^3+2^4+7(y^3-3y^2 2+3y\cdot 2^2-2^3)$
$+20(y^2+4-4y)+27y-54+15=0$

or, $y^4+(-8+7)y^3+(24-42+20)y^2+(-32+84-80+27)y$
$+(16-56-54+15+80)=0$

or, $y^4-y^3+2y^2-y+1=0$

or, ~~$y(y^3-y^2-19y-1)=0$~~

or, $(y^4+1)-(y^3+y)+2y^2=0$

or, $(y^2+\frac{1}{y^2})-(y+\frac{1}{y})+2=0$

or, $\{(y+\frac{1}{y})^2-2\cdot y\cdot\frac{1}{y}\}-(y+\frac{1}{y})+2=0$

or, $t^2-t=0$

or, $t(t-1)=0$

$t=0$, $t=1$,

$y+\frac{1}{y}=0$ $\quad$ $y+\frac{1}{y}=1$

or, $y=\pm i$ $\quad$ or, $y=\frac{1\pm\sqrt{1-4.1.1}}{2}$

$x=(\pm i-2)$ $\quad$ $x=(\frac{1\pm\sqrt{3}i}{2}-2)$

$=(-2\pm i)$ $\quad$ $=(\frac{-3\pm\sqrt{3}i}{2})$ A.

Theorem:- Let α be a root of the equ^n $x^n-1=0$. Then α^m is also a root of the equ^n $x^n-1=0$. where m is any integer.

Pf:- Since α be a root of the equ^n $x^n-1=0$,

$\therefore$ $\alpha^n-1=0$ or $\alpha^n=1$,

Now, $(\alpha^m)^n-1$
$=(\alpha^n)^m-1=(1)^m-1=1-1=0$.

$\therefore$ α^m is a root of the equ^n $x^n-1=0$ [Proved]

Theorem:- Let $\gcd(m,n)=1$ or m and n are relatively prime then the equ^{n}s $x^m-1=0$ and $x^n-1=0$ have no common root other than 1.

Pf:- Since $\gcd(m,n)=1$,
then there exist integers u and v such that $mu+nv=1$.
Let α be a common root of $x^m-1=0$ and $x^n-1=0$,

$\therefore$ $\alpha^m-1=0$ & $\alpha^n-1=0$

i.e $\alpha^m=1$ or $\alpha^n=1$

or, $(\alpha+2)(\alpha^2+2)\cdots(\alpha^{10}+2) = \dfrac{2^{11}+1}{3}$ [Proved]

11.(ii) Given equⁿ is $x^3+2x^2+1=0$ ① $\to \alpha,\beta,\gamma$,

$\Sigma\alpha=-2$, $\Sigma\alpha\beta=0$, $\alpha\beta\gamma=-1$.

Let $y=\beta\gamma$ or, $y=\dfrac{\alpha\beta\gamma}{\alpha}=-\dfrac{1}{\alpha}$ or, $\alpha=-\dfrac{1}{y}$,

$\left(-\dfrac{1}{y}\right)^3+2\left(-\dfrac{1}{y}\right)^2+1=0$

or, $y^3+2y-1=0$

Putting y by x, $x^3+2x-1=0$ ②.

$\therefore (x^3+2x^2+1)(x^3+2x-1)=(x-\alpha)(x-\beta)(x-\gamma)(x-\beta\gamma)(x-\gamma\alpha)(x-\alpha\beta)$

or, $x^6-1+2x^5+2x+2x^4-2x^2+4x^3=\{x^2-(\alpha+\beta\gamma)x+\alpha\beta\gamma\}$
$\{x^2-(\beta+\alpha\gamma)x+\alpha\beta\gamma\}\{x^2-(\gamma+\alpha\beta)x+\alpha\beta\gamma\}$

or, $\left\{\left(x-\frac{1}{x}\right)^3+3\left(x-\frac{1}{x}\right)\right\}+2\left\{\left(x-\frac{1}{x}\right)^2+2.x.\frac{1}{x}\right\}+2\left(x-\frac{1}{x}\right)+4$
$=\left\{\left(x-\frac{1}{x}\right)-(\alpha+\beta\gamma)\right\}\left\{\left(x-\frac{1}{x}\right)-(\beta+\alpha\gamma)\right\}\left\{\left(x-\frac{1}{x}\right)-(\gamma+\alpha\beta)\right\}$

[Dividing both side by x^3]

or, $t^3+3t+2t^2+4+2t+4=\{t-(\alpha+\beta\gamma)\}\{t-(\beta+\alpha\gamma)\}$
$\{t-(\gamma+\alpha\beta)\}$

or, $t^3+2t^2+5t+8=\{t-(\alpha+\beta\gamma)\}\{t-(\beta+\alpha\gamma)\}\{t-(\gamma+\alpha\beta)\}$

[Putting $x-\frac{1}{x}$ by t]

$\therefore$ $\alpha+\beta\gamma$, $\beta+\alpha\gamma$, $\gamma+\alpha\beta$ are the roots of the equⁿ
$x^3+2t^2+5x+8=0$. A.

(iii) Given equⁿ is $x^3+2x^2+1=0$ ① $\to\alpha,\beta,\gamma$,

$\Sigma\alpha=-2$, $\Sigma\alpha\beta=0$, $\alpha\beta\gamma=-1$,

Let $y=2\alpha$, or, $\alpha=\dfrac{y}{2}$

$\left(\dfrac{y}{2}\right)^3+2\left(\dfrac{y}{2}\right)^2+1=0$

or, $y^3+4y^2+8=0$ ②

Putting y by x we get, $x^3+4x^2+8=0$.

Now, $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ are the roots of the equⁿ $x^3+2x+1=0$

$\therefore (x^3+4x^2+8)(x^3+2x+1)=(x-2\alpha)(x-2\beta)(x-2\gamma)\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\left(x-\frac{1}{\gamma}\right)$

or, $x^6+4x^5+2x^4+17x^3+4x^2+16x+8=(x-2\alpha)\left(x-\frac{1}{\alpha}\right)(x-2\beta)\left(x-\frac{1}{\beta}\right)$
$(x-2\gamma)\left(x-\frac{1}{\gamma}\right)$

or, $\left\{x^3+\left(\frac{2}{x}\right)^3\right\}+4\left\{x^2+\left(\frac{2}{x}\right)^2\right\}+2\left\{x+\frac{2}{x}\right\}+17=\left\{\left(x+\frac{2}{x}\right)-\left(2\alpha+\frac{1}{\alpha}\right)\right\}$
$\left\{\left(x+\frac{2}{x}\right)-\left(2\beta+\frac{1}{\beta}\right)\right\}\left\{\left(x+\frac{2}{x}\right)-\left(2\gamma+\frac{1}{\gamma}\right)\right\}$

or, $\left(x+\frac{2}{x}\right)\left\{x^2+\left(\frac{2}{x}\right)^2-\cancel{2}\cdot\frac{2}{x}x\right\}+4\left\{\left(x+\frac{2}{x}\right)^2-2x\cdot\frac{2}{x}\right\}+2\left\{\left(x+\frac{2}{x}\right)\right\}+17$
$=\left\{\left(x+\frac{2}{x}\right)-\left(2\alpha+\frac{1}{\alpha}\right)\right\}\left\{\left(x+\frac{2}{x}\right)-\left(2\beta+\frac{1}{\beta}\right)\right\}\left\{\left(x+\frac{2}{x}\right)-\left(2\gamma+\frac{1}{\gamma}\right)\right\}$

or, $\cancel{t\{t+}\left(x+\frac{2}{x}\right)\left\{\left(x+\frac{2}{x}\right)^2-2\cdot x\cdot\frac{2}{x}-4\right\}+4\left\{\left(x+\frac{2}{x}\right)^2-4\right\}$
$+2\left\{\left(x+\frac{2}{x}\right)\right\}+17=\left\{\left(x+\frac{2}{x}\right)-\left(2\alpha+\frac{1}{\alpha}\right)\right\}\left\{\left(x+\frac{2}{x}\right)-\left(2\beta+\frac{1}{\beta}\right)\right\}$
$\left\{\left(x+\frac{2}{x}\right)-\left(2\gamma+\frac{1}{\gamma}\right)\right\}$

or, $t\{t^2-6\}+4\{t^2-4\}+2t+17=\left\{t-\left(2\alpha+\frac{1}{\alpha}\right)\right\}\left\{t-\left(2\beta+\frac{1}{\beta}\right)\right\}\left\{t-\left(2\gamma+\frac{1}{\gamma}\right)\right\}$

or, $t^3-6t+4t^2-16+2t+17=\left\{t-\left(2\alpha+\frac{1}{\alpha}\right)\right\}\left\{t-\left(2\beta+\frac{1}{\beta}\right)\right\}\left\{t-\left(2\gamma+\frac{1}{\gamma}\right)\right\}$

or, $t^3+4t^2-4t+1=\left\{t-\left(2\alpha+\frac{1}{\alpha}\right)\right\}\left\{t-\left(2\beta+\frac{1}{\beta}\right)\right\}\left\{t-\left(2\gamma+\frac{1}{\gamma}\right)\right\}$

$\therefore$ $2\alpha+\frac{1}{\alpha}$, $2\beta+\frac{1}{\beta}$, $2\gamma+\frac{1}{\gamma}$ are the roots of the equn

$t^3+4t^2-4t+1=0$ i.e $x^3+4x^2-4x+1=0$

13. Given equn is $x^4+3x^2+x+1=0$ (1)

From the relation between roots & co-efficient we get,

(1) $\Sigma\alpha=0$, $\Sigma\alpha\beta=3$, $\Sigma\alpha\beta\gamma=-1$, $\alpha\beta\gamma\delta=1$.

Now, $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ & $\frac{1}{\delta}$ are the roots of the equn,

$\left(\frac{1}{x}\right)^4+\left(\frac{3}{x}\right)^2+\left(\frac{1}{x}\right)+1=0$

or, $x^4+x^3+3x^2+1=0$

Now,
$(x^4+x^3+3x^2+1)(x^4+3x^2+x+1)=(x-\alpha)\left(x-\frac{1}{\alpha}\right)(x-\beta)\left(x-\frac{1}{\beta}\right)(x-\delta)\left(x-\frac{1}{\delta}\right)$
$(x-\gamma)\left(x-\frac{1}{\gamma}\right)$

or, $x^8+3x^6+x^5+x^4+x^7+3x^5+x^4+x^3+3x^6+9x^4+3x^3+3x^2+x^4+3x^2+x+1$
$=\left\{x^2-\left(\alpha+\frac{1}{\alpha}\right)x+1\right\}\left\{x^2-\left(\beta+\frac{1}{\beta}\right)x+1\right\}\left\{x^2-\left(\delta+\frac{1}{\delta}\right)x+1\right\}\left\{x^2-\left(\gamma+\frac{1}{\gamma}\right)x+1\right\}$

or, $x^8+x^7+6x^6+4x^5+12x^4+4x^3+6x^2+x+1=\left\{x^2-\left(\alpha+\frac{1}{\alpha}\right)x+1\right\}\left\{x^2-\left(\beta+\frac{1}{\beta}\right)x+1\right\}$
$\left\{x^2-\left(\gamma+\frac{1}{\gamma}\right)x+1\right\}\left\{x^2-\left(\delta+\frac{1}{\delta}\right)x+1\right\}$

or, $\left(x^4+\frac{1}{x^4}\right)+\left(x^3+\frac{1}{x^3}\right)+6\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)+12=\frac{1}{x^4}\left\{x^2-\left(\alpha+\frac{1}{\alpha}\right)x+1\right\}$
$\left\{x^2-\left(\beta+\frac{1}{\beta}\right)x+1\right\}\left\{x^2-\left(\gamma+\frac{1}{\gamma}\right)x+1\right\}\left\{x^2-\left(\delta+\frac{1}{\delta}\right)x+1\right\}$

or, $\left(x+\frac{1}{x}\right)^4+4-4\left(x+\frac{1}{x}\right)^2-2+\left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right)+6\left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)$
$=\left\{\left(x+\frac{1}{x}\right)-\left(\alpha+\frac{1}{\alpha}\right)\right\}\left\{\left(x+\frac{1}{x}\right)-\left(\beta+\frac{1}{\beta}\right)\right\}\left\{\left(x+\frac{1}{x}\right)-\left(\gamma+\frac{1}{\gamma}\right)\right\}\left\{\left(x+\frac{1}{x}\right)-\left(\delta+\frac{1}{\delta}\right)\right\}$

or, $t^4+t^3+2t^2+t+2=\left\{t-\left(\alpha+\frac{1}{\alpha}\right)\right\}\left\{t-\left(\beta+\frac{1}{\beta}\right)\right\}\left\{t-\left(\gamma+\frac{1}{\gamma}\right)\right\}\left\{t-\left(\delta+\frac{1}{\delta}\right)\right\}$

$\therefore$ $\alpha+\frac{1}{\alpha}$, $\beta+\frac{1}{\beta}$, $\gamma+\frac{1}{\gamma}$ are the roots of the equn

$x^4+x^3+3x^2+x+2=0$