5. THEORY OF EQUATIONS

Fundamental theorem of classical algebra : Every algebraic equation has a root, real or complex.

Theorem : An algebraic equation of degree n has n roots and no more.

Theorem : If α be a multiple root of the polynomial equation $f(x)=0$, of order r, then α is a multiple root of the equation $f'(x)=0$ of order $(r-1)$.

*** Note : $f(x)$ এবং $f'(x)$ এর Highest common factor (গ.সা.গু) যদি (i) $(x-\alpha)^{r-1}$ তাহলে $f(x)=0$ equation এর α একটি multiple root হবে of order r.

(ii) $(x-\alpha)^{r-1}(x-\beta)^{p-1}$ হয়, তাহলে, α multiple root of order r এবং β multiple root of p.

Exercise - 5 A

1. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 - x^3 + 2x^2 + x + 1 = 0$ find the value of

(i) $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$

(ii) $(2\alpha+1)(2\beta+1)(2\gamma+1)(2\delta+1)$

(iii) $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1)$.

(iv) $(\alpha^3+1)(\beta^3+1)(\gamma^3+1)(\delta^3+1)$.

A :- (i) Let $f(x) = x^4 - x^3 + 2x^2 + x + 1 = 0$ — (1)

since $\alpha, \beta, \gamma, \delta$ are the roots of the equation (1)

we have $x^4 - x^3 + 2x^2 + x + 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ — (2)

Putting $x=-1$

$1+1+2-1+1 = (-1)^4(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$

or, $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = 4$. A

(iii) Since $\alpha, \beta, \gamma, \delta$ are the roots of the equation

we have $(x-\alpha)(x-\beta)(x-\delta)(x-\gamma) = x^4 - x^3 + 2x^2 + x + 1$

$= (x^4+2x^2+1) - (x^3-x)$ — (2)

or, $(\alpha-x)(\beta-x)(\delta-x)(\gamma-x) = $

** [পাশাপাশি → 2 degree difference রাখিতে এক্ষেত্রে নেবে।]

(ii) Putting $x = -1/2$

$(-\frac{1}{2})^4 - (-\frac{1}{2})^3 + 2(-\frac{1}{2})^2 + (-\frac{1}{2}) + 1 = (-\frac{1}{2}-\alpha)(-\frac{1}{2}-\beta)(-\frac{1}{2}-\gamma)(-\frac{1}{2}-\delta)$

or, $\frac{1}{2^4} + \frac{1}{2^3} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + 1 = (-1)^4(\frac{1}{2})^4(2\alpha+1)(2\beta+1)(2\gamma+1)(2\delta+1)$

or, $\frac{1+2+2^4}{2^4} = \frac{1}{2^4}(2\alpha+1)(2\beta+1)(2\gamma+1)(2\delta+1)$

or, $(2\alpha+1)(2\beta+1)(2\gamma+1)(2\delta+1) = 3+2^4 = 19$. Ans

(iii) অনুরূপভাবে

Putting $x = i$

$(\alpha-i)(\beta-i)(\gamma-i)(\delta-i) = (1-2+1)-(i-i) = 0+2i$ — (3)

Putting $x = -i$

$(\alpha+i)(\beta+i)(\gamma+i)(\delta+i) = (1-2+1)-(2i) = 0-2i$ — (4)

Multiplying (3) & (4)

$(\alpha+i)(\alpha-i)(\beta+i)(\beta-i)(\gamma+i)(\gamma-i)(\delta+i)(\delta-i) = (0+2i)(0-2i)$

or, $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1) = 4$. Ans

(iv) [N.B:- $\alpha^3+1 = (\alpha+1)(\alpha^2-\alpha+1)$
$= (\alpha+1)(\alpha+w)(\alpha+w^2)$

$x^3+y^3+z^3-3xyz$
$= (x+y+z)(x+w^2y+wz)(x+wy+w^2z)$]

Since $\alpha, \beta, \gamma, \delta$ are the roots of the equation (1) we have,

$(x-\alpha)(x-\beta)(x-\delta)(x-\gamma) = x^4 - x^3 + 2x^2 + x + 1$

or, $(\alpha-x)(\beta-x)(\delta-x)(\gamma-x) = x(x^3+1) - (x^3-1) + 2x^2$ — (2)

Putting $x = -1$

$(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = (-1)(-1+1) - (-1-1) + 2$
$= 0+2+2 = 2+0+2$ (3)

Putting $x = -w$,

$(\alpha+w)(\beta+w)(\gamma+w)(\delta+w) = (-w)(-w^3+1) - (-w^3-1) + 2w^2$
$= 0+2+2w^2 = 2+0.w+2.w^2$ (4)

Putting $x = -w^2$

$(\alpha+w^2)(\beta+w^2)(\gamma+w^2)(\delta+w^2) = 0+2+2w = 2+0.w^2+2w$ (5)

Multiplying (3), (4) & (5)

$(\alpha^3+1)(\beta^3+1)(\gamma^3+1)(\delta^3+1) = (2)^3+(0)^3+(2)^3-3.2.0.2$
$= 8+8 = 16$. Ans

2. If $\alpha_1, \alpha_2, \dots \alpha_n$ be the roots of the equation $x^n + nax + b = 0$ prove that $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \dots (\alpha_1 - \alpha_n) = n(\alpha_1^{n-1} + a)$.

A:– Since $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ are the roots of the equation $x^n + nax + b = 0$ we can write,

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n) = x^n + nax + b \quad -(1)$$

Diff with respect to x,

$$(x-\alpha_2)\dots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_3)\dots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) + \dots + (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{n-1})$$
$$= nx^{n-1} + na \quad -(2)$$

Putting $x = \alpha_1$ in (2)

we get,

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n) = n\alpha_1^{n-1} + na = n(\alpha_1^{n-1} + a)$$

[Proved]

3. Solve the equation, given that it has multiple roots,

(i) $x^4 + 2x^3 + 2x^2 + 2x + 1 = 0$

A:– Let $f(x) = x^4 + 2x^3 + 2x^2 + 2x + 1$

Diff w. r. to x,

$$f'(x) = 4x^3 + 6x^2 + 4x + 2 = 2(2x^3 + 3x^2 + 2x + 1).$$

Now, we find highest common factor of $(x^4 + 2x^3 + 2x^2 + 2x + 1)$ and $(2x^3 + 3x^2 + 2x + 1)$.

$2x^3 + 3x^2 + 2x + 1 \,)\, x^4 + 2x^3 + 2x^2 + 2x + 1 \,(\, x + 1$

$\times 2$

$2x^4 + 4x^3 + 4x^2 + 4x + 2$

$2x^4 + 3x^3 + 2x^2 + x$

$x^3 + 2x^2 + 3x + 2$

$\times 2$

$2x^3 + 4x^2 + 6x + 4$

$2x^3 + 3x^2 + 2x + 1$

$x^2 + 4x + 3$

$x^2 + 4x + 3 \,)\, 2x^3 + 3x^2 + 2x + 1 \,(\, 2x - 5$

$2x^3 + 8x^2 + 6x$

$-5x^2 - 4x + 1$

$-5x^2 - 20x - 15$

$16x + 16 = 16(x+1)$

$x + 1 \,)\, x^2 + 4x + 3 \,(\, x + 3$

$x^2 + x$

$3x + 3$

$3x + 3$

$\times$

$\therefore$ Highest common factor is $(x+1)$

So, $x = -1$ is a multiple root of order $(1+1) = 2$, A

$\therefore (x+1)^2 = x^2+2x+1$ is a factor of $f(x)$.

$$x^2+2x+1 \overline{)\, x^4 + 2x^3 + 2x^2 + 2x + 1\,} (x^2+1$$
$$\underline{\cancel{x^4} + \cancel{2x^3} + x^2}$$
$$x^2 + 2x + 1$$
$$\underline{x^2 + 2x + 1}$$
$$\times$$

$\therefore$ $-1, -1$ are two roots of $f(x)=0$,

Other roots of $f(x)=0$, are obtain from $x^2+1=0$,

$\therefore$ The other roots are, $x=i$, $x=-i$.

$\therefore$ Roots of the given eqn^n are $-1, -1, i$ & $-i$ Ans

4. If α be a double root of the equation $x^n + P_1 x^{n-1} + \cdots + P_n$:
Prove that α is also a root of the equation,

$$P_1 x^{n-1} + 2P_2 x^{n-2} + 3P_3 x^{n-3} + \cdots + nP_n = 0,$$

Sol:- Let $f(x) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \cdots + P_{n-1}x + P_n$

Diff w.r.to x,

$$f'(x) = nx^{n-1} + P_1(n-1)x^{n-2} + \cdots + 2P_{n-2}x + P_{n-1}$$

Since, α is a double root of

$f(x)=0$ we have; $f(\alpha)=0$, $f'(\alpha)=0$

$f(\alpha)=0$ gives,

$$\alpha^n + P_1\alpha^{n-1} + P_2\alpha^{n-2} + \cdots + P_{n-1}\alpha + P_n = 0 \quad \text{—(1)} \times n$$

$f'(\alpha)=0$ gives,

$$n\alpha^{n-1} + (n-1)P_1\alpha^{n-2} + \cdots + 2P_{n-2}\alpha + P_{n-1} = 0 \quad (2) \times \alpha$$

$n \times (1)$ gives

$$\cancel{n\alpha^n} + nP_1\alpha^{n-1} + nP_2\alpha^{n-2} + \cdots + nP_{n-1}\alpha + nP_n = 0$$

$\alpha \times (2)$ gives

$$\underline{\cancel{n\alpha^n} + (n-1)P_1\alpha^{n-1} + (n-2)P_2\alpha^{n-2} + \cdots + P_{n-1}\alpha = 0}$$

$$P_1\alpha^{n-1} + 2P_2\alpha^{n-2} + 3P_3\alpha^{n-3} + \cdots + (n-1)P_{n-1}\alpha + nP_n = 0$$

$\therefore$ α is a root of the equation,

$$P_1 x^{n-1} + 2P_2 x^{n-2} + 3P_3 x^{n-3} + \cdots + nP_n = 0 \quad \text{[Proved]}$$

3. (ii) Solve the eqn, given that it has multiple roots.

$x^5+3x^4+5x^3+5x^2+3x+1=0$

A:– Let $f(x)=x^5+3x^4+5x^3+5x^2+3x+1$

Diff with respect to x

$f'(x)=5x^4+12x^3+15x^2+10x+3$

Now we find h.c.f of $x^5+3x^4+5x^3+5x^2+3x+1$ and $5x^4+12x^3+15x^2+10x+3$

$$
\begin{array}{l|l}
5x^4+12x^3+15x^2+10x+3 & x^5+3x^4+5x^3+5x^2+3x+1 \;\; (x+3 \\
 & \times 5 \\
 & 5x^5+15x^4+25x^3+25x^2+15x+5 \\
 & - (5x^5+12x^4+15x^3+10x^2+3x) \\
 & 3x^4+10x^3+15x^2+12x+5 \\
 & \times 5 \\
 & 15x^4+50x^3+75x^2+60x+25 \\
 & - (15x^4+36x^3+45x^2+30x+9) \\
 & 14x^3+30x^2+30x+16 \\
 & =2(7x^3+15x^2+15x+8)
\end{array}
$$

$$
\begin{array}{l|l}
7x^3+15x^2+15x+8 & 5x^4+12x^3+15x^2+10x+3 \;\; (5x+9 \\
 & \times 7 \\
 & 35x^4+84x^3+105x^2+70x+21 \\
 & - (35x^4+75x^3+75x^2+40x) \\
 & 9x^3+30x^2+30x+21 \\
 & \times 7 \\
 & 63x^3+210x^2+210x+147 \\
 & - (63x^3+135x^2+135x+72) \\
 & 75x^2+75x+75 \\
 & =75(x^2+x+1)
\end{array}
$$

$$
\begin{array}{l|l}
x^2+x+1 & 7x^3+15x^2+15x+8 \;\; (7x+8 \\
 & - (7x^3+7x^2+7x) \\
 & 8x^2+8x+8 \\
 & 8x^2+8x+8 \\
\end{array}
$$

∴ Highest common factor is (x^2+x+1).

So, $x=\omega$ and ω^2 are the multiple roots of order 2.

$(x^2+x+1)^2=\{(x^2+1)+x\}^2=(x^4+2x^3+3x^2+2x+1)$

$$
\begin{array}{l|l}
x^4+2x^3+3x^2+2x+1 & x^5+3x^4+5x^3+5x^2+3x+1 \;\; (x+1 \\
 & - (x^5+2x^4+3x^3+2x^2+x) \\
 & x^4+2x^3+3x^2+2x+1 \\
 & x^4+2x^3+3x^2+2x+1 \\
\end{array}
$$

∴ $\omega, \omega^2, \omega, \omega^2$ are four roots of $f(x)=0$.

∴ other root of $f(x)=0$ is obtain from $x+1=0$

6. Find the values of k for which the equation $x^3 - 9x^2 + 24x + k = 0$ may have multiple roots and solve the eqn in each case.

Ans:- Let $f(x) = x^3 - 9x^2 + 24x + k$ — (1)

$f'(x) = 3x^2 - 18x + 24$ — (2)

Now, $f'(x) = 0$ gives

$3x^2 - 18x + 24 = 0$

Or, $(x-4)(x-2) = 0$

$x = 4, 2$

The eqn $f(x) = 0$ has multiple root either $x = 2$ or $x = 4$ when $x = 2$ is a multiple root

$f(2) = 0$

Then

$8 - 9.4 + 24.2 + k = 0$ $\therefore k = -20$

Then the eqn becomes $(x^3 - 9x^2 + 24x - 20) = 0$

Or, $x^3 - 2x^2 - 7x^2 + 14x + 10x - 20 = 0$

Or, $x^2(x-2) - 7x(x-2) + 10(x-2) = 0$

Or, $(x-2)(x^2 - 7x + 10) = 0$

Or, $(x-2)(x-2)(x-5) = 0$

$\therefore$ The roots of the eqn are $2, 2, 5$

when $x = 4$ is a multiple root $f(4) = 0$,

$(4)^3 - 9(4)^2 + 24(4) + k = 0$

Or, $64 - 144 + 48 + k = 0$

$\therefore$ $k = -32/2$ $= -16$

Then the eqn becomes $(x^3 - 9x^2 + 24x + 16) = 0$

Or, $x^3 - 4x^2 - 5x^2 + 20x + 4x + 16 = 0$

Or, $x^2(x-4) - 5x(x-4) + 4(x-4) = 0$

Or, $(x-4)(x^2 - 5x + 4) = 0$

Or, $(x-4)\{x^2 - 4x - x + 4\} = 0$

Or, $(x-4)\{x(x-4) - 1(x-4)\} = 0$

Or, $(x-4)(x-4)(x-1) = 0$

$\therefore$ The roots of the eqn are $4, 4, 1$ Ans

8. Prove that 1 is a multiple root of the equⁿ $x^5 - 5x^3 + 5x^2 - 1 = 0$, find its order and solve the equⁿ.

Ans:– Let $f(x) = x^5 - 5x^3 + 5x^2 - 1$

$f'(x) = 5x^4 - 15x^2 + 10x$

$f''(x) = 20x^3 - 30x + 10$

$f'''(x) = 60x^2 - 30$

$\therefore$ Now $f(1) = 0$, $f'(1) = 0$, $f''(1) = 0$, $f'''(1) \neq 0$

$\therefore$ 1 is a multiple root of order 3.

$\therefore (x-1)^3 = x^3 - 3x^2 + 3x - 1$ is a factor of $f(x)$.

We divide $f(x)$ by $(x-1)^3$ by the method of detached co-efficient.

$$
\begin{array}{rrrr|rrrrrr}
1 & -3 & 3 & -1 & 1 & 0 & -5 & 5 & 0 & -1 \\
 & & & & -1 & \mp 3 & -3 & \mp 1 & & \\
\hline
1 & 3 & 1 & & & 3 & -8 & 6 & 0 & -1 \\
 & & & & & -3 & \mp 9 & -9 & \mp 3 & \\
\hline
 & & & & & & 1 & -3 & 3 & -1 \\
 & & & & & & 1 & -3 & 3 & -1 \\
\hline
\end{array}
$$

$\therefore f(x) = (x-1)^3 (x^2 + 3x + 1)$

Now, $x^2 + 3x + 1 = 0$ gives $x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$

$\therefore$ The roots of the equⁿ are $1, 1, 1, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$. Ans

9. [N.B – $f(x) = 0$ equⁿ এর co-efficient গুলি যদি Real হয় তাহলে $(\alpha + i\beta)$ যদি একটি Root হয় তবে $(\alpha - i\beta)$ -ও একটি Root হবে, যেখানে, α, β real]

(i) Solve the equation

(i) $x^4 - x^3 + 2x^2 - 2x + 4 = 0$, one root being $(1+i)$;

Ans:– The given equⁿ $x^4 - x^3 + 2x^2 - 2x + 4 = 0$ is an equⁿ with real co-efficient.

$\therefore$ Since $(1+i)$ is a root, $(1-i)$ is also a root of the given equⁿ.

$\therefore (x-1-i)(x-1+i) = (x-1)^2 + 1 = x^2 - 2x + 2$ is a factor of $x^4 - x^3 + 2x^2 - 2x + 4 = f(x)$.

We divide $f(x)$ by (x^2-2x+2) by the method of detached co-efficient.

$$\begin{array}{ccc|ccccc}
1 & -2 & 2 & 1 & -1 & 2 & -2 & 4 \\
 & & & -1 & \mp 2 & -2 & & \\
\hline
1 & 1 & 2 & & 1 & 0 & -2 & 4 \\
 & & & & -1 & \mp 2 & \pm 2 & \\
\hline
 & & & & & 2 & -4 & 4 \\
 & & & & & 2 & -4 & 4 \\
\hline
\end{array}$$

$\therefore f(x) = x^4 - x^3 + 2x^2 - 2x + 4$
$= (x^2-2x+2)(x^2+x+2)$

Other factor of $f(x)$ is (x^2+x+2).

Now $x^2+x+2=0$ gives $x = \dfrac{-1 \pm \sqrt{1-8}}{2} = \dfrac{-1 \pm i\sqrt{7}}{2}$,

$\therefore$ The roots of the equ^n are $(1+i), (1-i), \dfrac{-1 \pm i\sqrt{7}}{2}$ Ans.

(ii) $x^5 - 4x^4 + 5x^3 + x^2 - 4x + 5 = 0$, one root being $(2+i)$;

Ans:— The given equ^n $x^5 - 4x^4 + 5x^3 + x^2 - 4x + 5 = 0$ is an equ^n with real co-efficient.

Since $(2+i)$ is a root, $(2-i)$ is also a root of the given equ^n

$\therefore (x-2-i)(x-2+i) = (x-2)^2 + 1 = x^2 - 4x + 5$ is factor of

$f(x) = x^5 - 4x^4 + 5x^3 + x^2 - 4x + 5$.

We divide $f(x)$ by x^2-4x+5 by the method of detached co-efficient.

$$\begin{array}{ccc|cccccc}
1 & -4 & 5 & 1 & -4 & 5 & 1 & -4 & 5 \\
 & & & 1 & -4 & 5 & & & \\
\hline
1\ 0\ 0 & & 1 & 0 & 0 & 1 & -4 & 5 & \\
 & & & & & 1 & -4 & 5 & \\
\hline
\end{array}$$

$f(x) = x^5 - 4x^4 + 5x^3 + x^2 - 4x + 5$
$= (x^2-4x+5)(x^3+1)$

$x^3 + 1 = (x+1)(x^2 - x + 1)$

$x = \dfrac{+1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$. Ans

(iii) $2x^4 - 3x^3 - 3x^2 - 3x - 1 = 0$, one root being $1+\sqrt{2}$;

Ans: The given equ^n is an equ^n with rational co-efficient.

Since $(1+\sqrt{2})$ is a root of the given equ^n $(1-\sqrt{2})$ is also a root of the given equ^n

$\therefore (x-1-\sqrt{2})(x-1+\sqrt{2}) = (x-1)^2-2 = x^2+1-2x-2 = (x^2-2x-1)$.
is a factor of $(2x^4-3x^3-3x^2-3x-1)$.
We divide $f(x)$ by the method of detached co-efficient.

1	−2	−1	2	−3	−3	−3	−1
			−2	∓4	∓2		
2	1	1		1	−1	−3	
				1	∓2	∓1	
					1	−2	−1
					1	−2	−1

$\therefore f(x) = (x^2-2x-1)(2x^2+x+1)$
$\therefore$ The other factor is $(2x^2+x+1)$.
Now, $2x^2+x+1 = 0$ gives

$$x = \frac{-1\pm\sqrt{1-8}}{2.2} = \frac{-1\pm i\sqrt{7}}{4}.$$

$\therefore$ The roots of the equ^n are $(1\pm\sqrt{2}), \left(\frac{-1\pm i\sqrt{7}}{4}\right)$. A

(iv) $x^6-x^5-8x^4+2x^3+21x^2-9x-54=0$, one root being $\sqrt{2}+i$;
A: since the given equ^n is an equ^n with rational co-efficients.
Since $(\sqrt{2}+i)$ is a root of the given equ^n $(\sqrt{2}-i)$, $(-\sqrt{2}+i)$, $(-\sqrt{2}-i)$ are also roots of the given equ^n.

$(x-\sqrt{2}-i)(x-\sqrt{2}+i)(x+\sqrt{2}-i)(x+\sqrt{2}+i)$
$= \{(x-\sqrt{2})^2+1\}\{(x+\sqrt{2})^2+1\}$
$= (x^2+3-2\sqrt{2}x)(x^2+3+2\sqrt{2}x)$
$= (x^2+3)^2-(2\sqrt{2}x)^2 = x^4+9+6x^2-8x^2 = x^4-2x^2+9$ is a factor of $f(x)$.

We divide $f(x)$ by the method of detached co-efficient.

1	0	−2	0	9	1	−1	−8	2	21	−9	−54
					−1	0	∓2	−0	−9		
1	−1	−6				−1	−6	2	12	−9	
						∓1	−0	2	0	∓9	
							−6	0	12	0	−54
							−6	0	12	0	−54
									×		

$f(x) = (x^4-2x^2+9)(x^2-x-6)$
$\therefore$ The other factor is (x^2-x-6).

Now, $x^2-x-6=0$
or, $(x-3)(x+2)=0$
$x = 3, -2$

$\therefore$ The roots of the equ^n are $+3, -2, \sqrt{2}\pm i, -\sqrt{2}\pm i$. A

(v) $x^4+x^3-2x+8=0$ having a complex root of modulus $\sqrt{2}$; solve the equⁿ;

A:— Since the given equⁿ is an equⁿ with real co-efficients having a complex root of modulus $\sqrt{2}$. We can take roots of the equⁿ are $\sqrt{2}(\cos\theta \pm i\sin\theta)$, α, β,

$$(x-\sqrt{2}\cos\theta - \sqrt{2}i\sin\theta)(x-\sqrt{2}\cos\theta+\sqrt{2}i\sin\theta)(x-\alpha)(x-\beta)$$

$$=\{(x-\sqrt{2}\cos\theta)^2+2\sin^2\theta\}\{x^2-(\alpha+\beta)x+\alpha\beta\}$$

$$=(x^2+2\cos^2\theta-2\sqrt{2}x\cos\theta+2\sin^2\theta)(x^2-(\alpha+\beta)x+\alpha\beta)$$

$$=x^4-x^3(\alpha+\beta+2\sqrt{2}\cos\theta)+x^2\{\alpha\beta+2\sqrt{2}(\alpha+\beta)\cos\theta+2\}$$

$+x\{-2\sqrt{2}\alpha\beta\cos\theta-2(\alpha+\beta)\}+2\alpha\beta$ is identical with x^4+x^3-2x+8

$\alpha+\beta+2\sqrt{2}\cos\theta=-1$ ①

$\alpha\beta+2\sqrt{2}(\alpha+\beta)\cos\theta+2=0$ ②

$-2\sqrt{2}\alpha\beta\cos\theta-2(\alpha+\beta)=-2$ ③

$2\alpha\beta=8$ ④

From ④

$\alpha\beta=4$

From ②

$4+2\sqrt{2}(\alpha+\beta)\cos\theta+2=0$ ⑤

from ③

Or, $-8\sqrt{2}\cos\theta-2(\alpha+\beta)=-2$ ⑥

From ① & ⑥

$\cos\theta=\pi/4$

From ⑤

$4+2\sqrt{2}(\alpha+\beta)\frac{1}{\sqrt{2}}=-6$

Or, $\alpha+\beta=-3$

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=-7$

Or, $\alpha-\beta=\pm i\sqrt{7}$

$\therefore \alpha=\frac{-3+i\sqrt{7}}{2}$, $\beta=\frac{-3-i\sqrt{7}}{2}$

$\therefore$ Roots of the equⁿ are $\sqrt{2}(\cos\pi/4 \pm i\sin\pi/4)$, $\frac{-3\pm i\sqrt{7}}{2}$

i.e $(1\pm i)$, $\left(\frac{-3\pm i\sqrt{7}}{2}\right)$ Ans

(vi) $3x^4+2x^3+9x^2+4x+6=0$ having a complex root of modulus 1. Solve the eqn.

A:– The given eqn is an eqn with real co-efficient, Having a complex root of modulus 1. We can take the roots of the eqn are $1(\cos\theta \pm i\sin\theta)$, α, β

$\therefore 3(x-\cos\theta-i\sin\theta)(x-\cos\theta+i\sin\theta)(x-\alpha)(x-\beta)$

$= 3x^4 + x^3\{-6\cos\theta - 3(\alpha+\beta)\} + x^2\{3\alpha\beta + 6(\alpha+\beta)\cos\theta + 3\}$

$+ x\{-6\alpha\beta\cos\theta - 3(\alpha+\beta)\} + 3\alpha\beta$

is identical with $3x^4+2x^3+9x^2+4x+6$

$-6\cos\theta - 3(\alpha+\beta) = 2$ ① $\quad 3\alpha\beta + 6(\alpha+\beta)\cos\theta + 3 = 9$ ②

$-6\alpha\beta\cos\theta - 3(\alpha+\beta) = 4$ ③ $\quad 3\alpha\beta = 6$ ④

From ④

$\alpha\beta = 2$

From ②

$6+3+6(\alpha+\beta)\cos\theta = 9$

or, $6(\alpha+\beta)\cos\theta = 0$

$-12\cos\theta - 3(\alpha+\beta) = 4$

$-12\cos\theta - 6(\alpha+\beta) = 4$

$3(\alpha+\beta) = 0$

$\alpha+\beta = 0$

From ①

$\cos\theta = -\frac{1}{3}$

$\theta = \cos^{-1}\left(-\frac{1}{3}\right)$

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{4}} = \frac{2\sqrt{2}}{3}$

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = 0 - 4.2 = -8$

$\alpha-\beta = 2\sqrt{2}i$

$\alpha+\beta+\alpha-\beta = 2\sqrt{2}i$ or, $\alpha = \sqrt{2}i$

$\beta = -\sqrt{2}i$

$\therefore$ The roots of the eqn are $\left(-\frac{1}{3} \pm \frac{i2\sqrt{2}}{3}\right)$, $(\sqrt{2}i)$, $(-\sqrt{2}i)$

10. From a biquadratic eqn with rational co-efficients two of whose roots are $(\sqrt{3}\pm 2)$.

A:– Since the required eqn is an biquadratic eqn with rational co-efficient.

Two of whose roots are $(\sqrt{3}\pm 2)$, other two roots are $(-\sqrt{3}+2)$, $(-\sqrt{3}-2)$.

$\therefore$ The required eqn is

$(x-\sqrt{3}-2)(x-\sqrt{3}+2)(x+\sqrt{3}-2)(x+\sqrt{3}+2) = 0$

or, $\{(x+2)^2-3\}\{(x-2)^2-3\} = 0$

or, $(x^2+4+4x-3)(x^2+4-4x-3)=0$

or, $(x^2+4x+1)(x^2-4x+1)=0$

or, $(x^2+1)^2-16x^2=0$

or, $x^4+1+2x^2-16x^2=0$

or, $x^4-14x^2+1=0$ Ans

11. From a biquadratic equⁿ with rational co-efficients two of the roots are $2i \pm 1$.

A:- The required equⁿ is an equⁿ with rational co-efficient (real co-efficient) and two of whose roots are $(2i \pm 1)$ all roots of the equⁿ are $(2i+1)$, $(2i-1)$, $(-2i+1)(-2i-1)$

∴ The required equⁿ is

$(x-2i-1)(x-2i+1)(x+2i-1)(x+2i+1)=0$

or, $\{(x-1)^2+4\}\{(x+1)^2+4\}=0$

or, $(x^2+1-2x+4)(x^2+2x+1+4)=0$

or, $\{(x^2+5)^2-4x^2\}=0$

or, $x^4+6x^2+25=0$ Ans

12. The equⁿ $3x^3+5x^2+5x+3=0$ has three distinct roots of equal moduli. Solve it.

A:- The equⁿ $3x^3+5x^2+5x+3=0$ (1) is an equⁿ with real co-efficient has three roots of equal moduli.

Let the roots of the equⁿ be r, $r(\cos\theta \pm i\sin\theta)$.

Therefore, where θ be a real number

$3(x-r)(x-r\cos\theta-ri\sin\theta)(x-r\cos\theta+ir\sin\theta)=0$

or, $3(x-r)(x^2-2xr\cos\theta+r^2)=0$

i.e $3x^3-6x^2r\cos\theta+3xr^2-3x^2r+6xr^2\cos\theta-3r^3=0$

i.e $3x^3-x^2(6r\cos\theta+3r)+x(3r^2+6r^2\cos\theta)-3r^3=0$

is identical with $3x^3+5x^2+5x+3=0$

$-(6r\cos\theta+3r)=5$ (2) $3r^2+6r^2\cos\theta=5$ (3)

$-3r^3=3$ (4)

from (1) $\qquad$ from (2)

$\therefore r^3 = -1 \qquad -\{6(-1)\cos\theta + 3(-r)\} = 5$

$r = -1 \qquad$ or, $6\cos\theta = 2$

$\cos\theta = \frac{1}{3}$

$\sin\theta = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$

$\therefore$ The roots of the eqn are $-1, -\left(\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}\right)$ i.e

$-1, \left(\frac{-1 \pm i2\sqrt{2}}{3}\right)$. Ans

13. The eqn $x^3 - x^2 + 3x - 27 = 0$ has three distinct roots of equal moduli. Solve it.

A:— The given eqn with real co-efficient has three roots of equal moduli.

Let the roots of the eqn be r, $r(\cos\theta \pm i\sin\theta)$ θ is real.

$(x-r)(x - r\cos\theta - ir\sin\theta)(x - r\cos\theta + ir\sin\theta) = 0$

or, $(x-r)\{(x - r\cos\theta)^2 + r^2\sin^2\theta\} = 0$

or, $(x-r)(x^2 + r^2\cos^2\theta - 2xr\cos\theta + r^2\sin^2\theta) = 0$

or, $(x-r)\{x^2 + r^2 - 2xr\cos\theta\} = 0$

or, $(x^3 + xr^2 - 2x^2r\cos\theta - x^2r - r^3 + 2xr^2\cos\theta = 0$

or, $x^3 + x^2\{-2r\cos\theta - r\} + x\{r^2 + 2r^2\cos\theta\} - r^3 = 0$

is identical with $x^3 - x^2 + 3x - 27 = 0$

$\cancel{-}(2r\cos\theta + r) = \cancel{-}1 \qquad r^2 + 2r^2\cos\theta = 3 \qquad r^3 = 27$

$r = 3$

or, $2r\cos\theta + r = 1$

or, $6\cos\theta + 3 = 1 \qquad \sin\theta = \sqrt{1-\left(-\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

or, $\cos\theta = -\frac{1}{3}$

$\therefore$ The roots of the given eqn are $3, 3\left(-\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}\right)$

i.e $3, (-1 \pm 2\sqrt{2}i)$. Ans

14. The equation $3x^4 + x^3 + 4x^2 + x + 3 = 0$ has four distinct roots of equal moduli.

A:— Let r be the modulus.

Two cases may arises.

(1) Two roots are real and two are complex.

Let the roots be $r, -r, r(\cos\theta \pm i\sin\theta)$.

(ii) All the roots are complex.
Let the roots of the eqn be $r(\cos\theta \pm i\sin\theta)$, $r(\cos\phi \pm i\sin\phi)$

Case-1
The given eqn is identical with

11 $3(x-r)(x+r)(x - r\cos\theta + ir\sin\theta)(x - r\cos\theta - ir\sin\theta) = 0$

Or, $3(x^4 - 2r\cos\theta\, x^3 + 2r^3 x\cos\theta - r^4) = 0$

Or, $3x^4 - 6r\cos\theta\, x^3 + 6r^3 x\cos\theta - 3r^4 = 0$

Since, there is a term containing x^2 in the given (Our assumption that the roots of the given eqn are (Case-1) two real and two imaginary is not true).

Case 1 is not true.

Case-2

$3(x - r\cos\theta - ir\sin\theta)(x - r\cos\theta + ir\sin\theta)(x - r\cos\phi - ir\sin\phi)$
$(x - r\cos\phi + ir\sin\phi) = 0$ is identical with

12 Or, $3\{(x - r\cos\theta)^2 + r^2\sin^2\theta\}\{(x - r\cos\phi)^2 + r^2\sin^2\phi\} = 0$

Or, $3(x^2 + r^2\cos^2\theta - 2xr\cos\theta + r^2\sin^2\theta)(x^2 + r^2\cos^2\phi - 2xr\cos\phi + r^2\sin^2\phi)$

Or, $3(x^2 + r^2 - 2xr\cos\theta)(x^2 + r^2 - 2xr\cos\phi) = 0$

Or, $3\{x^4 + x^2r^2 - 2x^3 r\cos\phi + x^2r^2 + r^4 - 2xr^3\cos\phi - 2x^3 r\cos\theta$
$- 2xr^3\cos\theta + 4x^2r^2\cos\theta\cos\phi\} = 0$

Or, $3x^4 - 6r(\cos\theta + \cos\phi)x^3 + 6r^2(1 + 2\cos\theta\cos\phi)x^2$
$- 6r^3(\cos\theta + \cos\phi)x + 3r^4 = 0$

$\therefore r(\cos\theta + \cos\phi) = -\frac{1}{6}$ (1)

$6r^2(1 + 2\cos\theta\cos\phi) = 4$

Or, $r^2(1 + 2\cos\theta\cos\phi) = \frac{2}{3}$ (2)

$r^3(\cos\theta + \cos\phi) = -\frac{1}{6}$ (3)

$3r^4 = 3$ (4)

From (4)

$r = 1$

From (1) & (3)

$\cos\theta + \cos\phi = -\frac{1}{6}$

$(\cos\theta - \cos\phi)^2 = (\cos\theta + \cos\phi)^2 - 4\cos\theta\cos\phi$

$= (-\frac{1}{6})^2 - 4(-\frac{1}{6})$ [From (6)]

$= \frac{25}{36}$

From (2)

$1 + 2\cos\theta\cos\phi = \frac{2}{3}$

or, $2\cos\theta\cos\phi = -\frac{1}{3}$ (6)

$\cos\theta + \cos\phi = -\frac{1}{6}$

$\cos\theta - \cos\phi = \frac{5}{6}$

$2\cos\theta = \frac{4}{6}$

or, $\cos\theta = \frac{1}{3}$

$\sin\theta = \frac{2\sqrt{2}}{3}$

from (1)

$\cos\phi = -\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$

$\sin\phi = \sqrt{1-(-\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$

∴ The required roots of the equ^n are

$(\frac{1}{3} \pm \frac{i2\sqrt{2}}{3})$, $(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2})$ ~~A~~

5. The equ^n $x^4 - 2x^3 + 18x^2 - 18x + 81 = 0$ has four distinct roots of equal moduli. Solve it.

A:— Let r be the modulus.

Two cases may arises.

(i) Two roots are real and two complex.

Let the roots be $r, -r, r(\cos\theta \pm i\sin\theta)$

(ii) All the roots are complex.

Let the roots of the equ^n be $r(\cos\theta \pm i\sin\theta), r(\cos\phi \pm i\sin\phi)$.

Case 1 :—

The given equ^n is identical with $x^4 - 2x^3 + 18x^2 - 18x + 81 = 0$

$(x-r)(x+r)(x - r\cos\theta - ir\sin\theta)(x - r\cos\theta + ir\sin\theta) = 0$

or, $(x^2 - r^2)(x^2 + r^2 - 2xr\cos\theta) = 0$

or, $x^4 - 2x^3 r\cos\theta + 2xr^3\cos\theta - r^4 = 0$

Since there is a term containing x^2 in the given equ^n

(our assumption that the roots of the given equ^n are two real and two imaginary is not correct).

Case 1 is not true.

Case 2 :—

The given equ^n is identical with

$(x - r\cos\theta - ir\sin\theta)(x - r\cos\theta + ir\sin\theta)(x - r\cos\phi + ir\sin\phi)$
$(x - r\cos\phi - ir\sin\phi) = 0$

or, $(x^2 + r^2 - 2xr\cos\theta)(x^2 + r^2 - 2xr\cos\phi) = 0$

or, $x^4 + x^2r^2 - 2x^3r\cos\phi + x^2r^2 + r^4 - 2xr^3\cos\phi - 2x^3r\cos\theta$
$- 2xr^3\cos\theta + 4x^2r^2\cos\theta\cos\phi = 0$

or, $x^4 + \{x^3(-2r\cos\phi - 2r\cos\theta)\} + \{x^2(2r^2 + 4r^2\cos\theta\cos\phi)\}$

$+ \{x(-2r^3\cos\theta - 2r^3\cos\phi)\} + r^4 = 0$

$-2r(\cos\theta + \cos\phi) = -2$ $\quad\quad 2r^2 + 4r^2\cos\theta\cos\phi = 18$

or, $\cos\theta + \cos\phi = \frac{1}{r}$ (1) $\quad\quad$ or, $2r^2(1 + 2\cos\theta\cos\phi) = 18$ (2)

$-2r^3(\cos\theta + \cos\phi) = -18$ $\quad\quad r^4 = 81$ (4)

or, $r^3(\cos\theta + \cos\phi) = 9$ (3)

from (4) $\quad r = 3$

from (2), $\quad 18(1 + 2\cos\theta\cos\phi) = 18$

or, $\quad 2\cos\theta\cos\phi = 0$

from (1)

$\cos\theta + \cos\phi = \frac{1}{3}$

or, $(\cos\theta - \cos\phi)^2 = \left(\frac{1}{3}\right)^2 - 4.0$

$= \frac{1}{9}$

or, $\cos\theta - \cos\phi = \frac{1}{3}$ (5)

From (1) & (5),

$\cos\theta + \cos\phi = \frac{1}{3}$

$\cos\theta - \cos\phi = \frac{1}{3}$

$2\cos\theta = \frac{2}{3}$

or, $\cos\theta = \frac{1}{3}$.

$\sin\theta = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$,

$\cos\phi = 0$

$\sin\phi = \sqrt{1-0} = 1$,

$\therefore$ The required roots of the given eqn are $3\left(\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}\right)$

i.e $(1 \pm i2\sqrt{2})$ and $3(0 \pm i)$ i.e $\pm 3i$ Ans.

16. (i) Prove that the roots of the equn are all real

$\frac{1}{x+a_1} + \frac{1}{x+a_2} + \cdots + \frac{1}{x+a_n} = \frac{1}{x}$, where $a_1, a_2, \cdots a_n$

are all +ve real numbers.

:- The given equn is

$\frac{1}{x+a_1} + \frac{1}{x+a_2} + \cdots + \frac{1}{x+a_n} = \frac{1}{x}$ (1)

or, $\frac{x}{x+a_1} + \frac{x}{x+a_2} + \cdots + \frac{x}{x+a_n} = \frac{x}{x}$

or, $\frac{x}{x+a_1} - 1 + \frac{x}{x+a_2} - 1 + \cdots + \frac{x}{x+a_n} - 1 = 1 - n$

or, $\frac{-a_1}{x+a_1} + \frac{-a_2}{x+a_2} + \cdots + \frac{-a_n}{x+a_n} = 1 - n$

or, $\frac{a_1}{x+a_1} + \frac{a_2}{x+a_2} + \cdots + \frac{a_n}{x+a_n} = n - 1$ (2)

Let $(\alpha + i\beta)$ be a root of the equn (2) where α, β are all real,

Since the co-efficients of the equn (1) i.e the co-efficients of (1) are

real, $(\alpha - i\beta)$ is also a root of the given equn (2)

$\frac{a_1}{\alpha + i\beta + a_1} + \frac{a_2}{\alpha + i\beta + a_2} + \cdots + \frac{a_n}{\alpha + i\beta + a_n} = n - 1$ (3)

and $\dfrac{a_1}{\alpha-i\beta+a_1}+\dfrac{a_2}{\alpha-i\beta+a_2}+\cdots+\dfrac{a_n}{\alpha-i\beta+a_n}=n-1$ (4)

Now (3) − (4) we get,

$$\left(\frac{\alpha-i\beta+a_1-\alpha-i\beta-a_1}{(\alpha+a_1)^2+\beta^2}\right)+a_2\left(\frac{\alpha-i\beta+a_2-\alpha-i\beta-a_2}{(\alpha+a_2)^2+\beta^2}\right)+\cdots+a_n\left(\frac{\alpha-i\beta+a_n-\alpha-i\beta-a_n}{(\alpha+a_n)^2+\beta^2}\right)=0$$

or, $-2i\beta\left[\dfrac{a_1}{(\alpha+a_1)^2+\beta^2}+\cdots+\dfrac{a_n}{(\alpha+a_n)^2+\beta^2}\right]=0$

Since $a_1, a_2, \ldots a_n$ are positive real numbers and also α, β are real, the bracketed expression $\neq 0$

$\therefore \beta=0$

$\therefore$ the roots of the given equ^n are all real. [Proved]

15. (ii) $\dfrac{1}{x+a_1}+\dfrac{1}{x+a_2}+\cdots+\dfrac{1}{x+a_n}=\dfrac{1}{x}$, where $a_1, a_2, \ldots a_n$ are all negative real numbers.

Ans:– $\dfrac{1}{x+a_1}+\dfrac{1}{x+a_2}+\cdots+\dfrac{1}{x+a_n}=\dfrac{1}{x}$

or, $\dfrac{x}{x+a_1}+\dfrac{x}{x+a_2}+\cdots+\dfrac{x}{x+a_n}=1$

or, $\dfrac{x}{x+a_1}-1+\dfrac{x}{x+a_2}-1+\cdots+\dfrac{x}{x+a_n}-1=1-n$

or, $\dfrac{a_1}{x+a_1}+\dfrac{a_2}{x+a_2}+\cdots+\dfrac{a_n}{x+a_n}=n-1$ (2)

Let $(\alpha+i\beta)$ be a root of the equ^n (2) where α, β are real. Since the co-efficients of equ^n (1) i.e. the co-efficients of equ^n (2) are real, $(\alpha-i\beta)$ is also a root of the equ^n (2).

$\dfrac{a_1}{\alpha+i\beta+a_1}+\dfrac{a_2}{\alpha+i\beta+a_2}+\cdots+\dfrac{a_n}{\alpha+i\beta+a_n}=n-1$ (3)

and $\dfrac{a_1}{\alpha-i\beta+a_1}+\dfrac{a_2}{\alpha-i\beta+a_2}+\cdots+\dfrac{a_n}{\alpha-i\beta+a_n}=n-1$ (4)

Now (3) − (4) gives,

$$\left(\frac{a_1}{\alpha+i\beta+a_1}-\frac{a_1}{\alpha-i\beta+a_1}\right)+\left(\frac{a_2}{\alpha+i\beta+a_2}-\frac{a_2}{\alpha-i\beta+a_2}\right)+\cdots+\left(\frac{a_n}{\alpha+i\beta+a_n}-\frac{a_n}{\alpha-i\beta+a_n}\right)=0$$

or, $2i\beta\left[\dfrac{-a_1}{(\alpha+a_1)^2+\beta^2}+\cdots+\dfrac{-a_n}{(\alpha+a_n)^2+\beta^2}\right]=0$

Since $a_1, a_2, \ldots a_n$ are all negative real number and α, β are all real, the bracketed expression $\neq 0$ and also α, β are real. The bracketed expression $\neq 0$ and also $\beta=0$

$\therefore$ The roots of the given equ^n are all real. [Proved]

17. If a polynomial eqn $f(x)=0$ with real co-efficients has a complex root $(\alpha+i\beta)^p$ where α, β are real and p is a positive integer, prove that $(\alpha-i\beta)^p$ also a root of the eqn.

De Moivre's theorem :—

Ans:— If θ is a real number and n is a positive integer then $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

Let $(\alpha+i\beta) = r(\cos\theta + i\sin\theta)$ [Polar representation of $(\alpha+i\beta)$]

where $r\cos\theta = \alpha$ $\quad r\sin\theta = \beta$

Since p is a positive integer

$(\alpha+i\beta)^p = \{r(\cos\theta + i\sin\theta)\}^p$

$= r^p(\cos p\theta + i\sin p\theta)$ [By De Moivre's theorem]

Since $f(x)=0$ is a polynomial eqn with real co efficient having a root $(\alpha+i\beta)^p = r^p(\cos p\theta + i\sin p\theta)$

$r^p(\cos p\theta - i\sin p\theta)$

$= r^p(\cos\theta - i\sin\theta)^p$

$= (\alpha-i\beta)^p$ is also a root of the given eqn. [Proved]

Theorem: Let $f(x)$ be a polynomial with real co-efficient. If α, β be two distinct real numbers such that $f(\alpha)$ and $f(\beta)$ are of opposite sings then the eqn $f(x)=0$ has at least one real root lying between α and β.

18. If a polynomial eqn $f(x)=0$ with real co-efficients has a complex root $(\alpha+i\beta)$ of multiplicity p, prove that $\alpha-i\beta$ is also a root of the eqn $f(x)=0$ of multiplicity p.

Ans:—

20. Prove that the roots of the eqn $(2x+3)(2x+1)(x-1)(4x-7)+(x+1)(2x-1)(2x-3)=0$ are all real and different. Separate the intervals in which the roots lie.

Ⓢ:— Let $f(x)=(2x+3)(2x+1)(x-1)(4x-7)+(x+1)(2x-1)(2x-3)$

$f(-1)=-22<0,\ f(\frac{1}{2})=20>0\quad f(\frac{3}{2})=-12<0$

$-\infty \quad -1 \quad 0 \quad \frac{1}{2} \quad \frac{3}{2} \quad \infty$

$f(-\infty)>0$ ve $\quad f(-1)<0$ $-ve$ $\quad f(1/2)>0$ $+ve$ $\quad f(3/2)<0$ $-ve$ $\quad f(\infty)>0$ $+ve$

There exists at least one real root of the eqn $f(x)=0$ in the intervals $(-\infty,-1),\ (-1,7\frac{1}{2}),\ (\frac{1}{2},\frac{3}{2}),\ (\frac{3}{2},\infty)$

Since the eqn of is of degree 4, all its roots are real and simple.

21. Show that the eqn $x^4-14x^2+24x+k=0$ has
(i) Four real and unequal roots if $-11<K<-8$,
(ii) two distinct real roots if $-8<k<117$,
(iii) no real roots if $k>117$.
Discuss the cases when $k=117$, $k=-8$ and $k=-11$.

Ⓢ:— (i) Let $f(x)=x^4-14x^2+24x+k$

$f'(x)=4x^3-28x+24$
$=4x^3-4x^2+4x^2-4x-24x+24$
$=4x^2(x-1)+4x(x-1)-24(x-1)$
$=(x-1)\,4\left(x^2+x-\frac{24}{4}\right)$
$=4(x-1)(x^2+x-6)$
$=4(x-1)(x+3)(x-2)$

$x=-3,1,2$

By Rolle's theorem, between two consecutive real roots of $f(x)=0$, there exists at least one real root of $f'(x)=0$.

$f(-\infty)>0\quad f(-3)=81-126-72+k=(k-117)$
$f(1)=1-14+24+k=(11+k)$

(iii) Solve the equ^n, given that it has multiple roots.

$x^6+2x^5+5x^4+6x^3+7x^2+4x+2=0$

A:- Let $f(x)= x^6+2x^5+5x^4+6x^3+7x^2+4x+2$

D.w.r.to x

$f'(x) = 6x^5+10x^4+20x^3+18x^2+14x+4$

Now we find highest common factor of $x^6+2x^5+5x^4+6x^3+7x^2+4x+2$ and $6x^5+10x^4+20x^3+18x^2+14x+4$.

$=2(3x^5+5x^4+10x^3+9x^2+7x+2)$.

$3x^5+5x^4+10x^3+9x^2+7x+2 \,)\, x^6+2x^5+5x^4+6x^3+7x^2+4x+2 \,(\, x+1$

$\quad 3$

$3x^6+6x^5+15x^4+18x^3+21x^2+12x+6$

$-\ 3x^6+5x^5+10x^4+9x^3+7x^2+2x$

$x^5+5x^4+9x^3+14x^2+10x+6$

$\quad 3$

$3x^5+15x^4+27x^3+42x^2+30x+18$

$-\ 3x^5+5x^4+10x^3+9x^2+7x+2$

$10x^4+17x^3+33x^2+23x+16$

$10x^4+17x^3+33x^2+23x+16 \,)\, 3x^5+5x^4+10x^3+9x^2+7x+2 \,(\, 3x+1$

$\quad 10$

$30x^5+50x^4+100x^3+90x^2+70x+20$

$-\ 30x^5+51x^4+99x^3+69x^2+48x$

$-x^4+x^3+21x^2+22x+20$

$\quad 10$

$-10x^4+10x^3+210x^2+220x+200$

$10x^4+17x^3+33x^2+23x+16$

$27x^3+243x^2+243x+216$

$=27(x^3+9x^2+9x+8)$

$x^3+9x^2+9x+8 \,)\, 10x^4+17x^3+33x^2+23x+16 \,(\, 10x+73$

$-\ 10x^4+90x^3+90x^2+80x$

$-73x^3-57x^2-57x+16$

$73x^3+657x^2+657x+584$

$600x^2+600x+600 = 600(x^2+x+1)$

$x^2+x+1 \,)\, x^3+9x^2+9x+8 \,(\, x+8$

$-\ x^3+x^2+x$

$8x^2+8x+8$

$8x^2+8x+8$

$\times$

x^2+x+1 is the highest common factor.