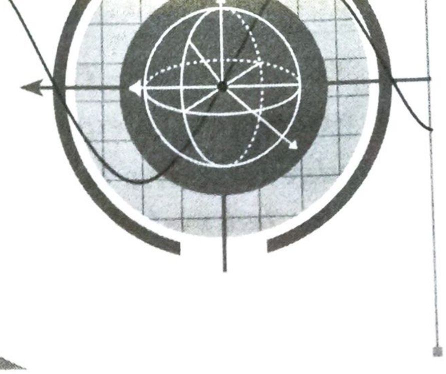
Scalar and Vector Triple Product



Introduction

We have seen that the cross product of two vectors $\vec{b}$ and $\vec{c}$ *i.e.*, $\vec{b} \times \vec{c}$ is a vector which is perpendicular to both $\vec{b}$ and $\vec{c}$ and its direction is such a way that $\vec{b}$, $\vec{c}$ and $\vec{b} \times \vec{c}$ form a right handed system. In this chapter we consider dot and cross product of $\vec{b} \times \vec{c}$ by another vector $\vec{a}$ from the left *i.e.*, $\vec{a} \cdot (\vec{b} \times \vec{c})$ and $\vec{a} \times (\vec{b} \times \vec{c})$. The first product, which is a scalar quantity is called scalar triple product and the second one, which is a vector quantity is called vector triple product.

2.1 Scalar Triple Product

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors. Then $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product or box product of the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and it is denoted also by $[\vec{a}\ \vec{b}\ \vec{c}]$ or $[\vec{a}, \vec{b}, \vec{c}]$.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, where $\hat{i}$, $\hat{j}$, $\hat{k}$ are three unit vectors along the axes.

$$\text{Now } \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)$$

$$\begin{aligned} \text{Now, } \vec{a} \cdot (\vec{b} \times \vec{c}) &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_2c_3 - b_3c_2)\hat{i} - (b_1c_3 - b_3c_1)\hat{j} + (b_1c_2 - b_2c_1)\hat{k} \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\ &\quad [\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \text{ and } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0] \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

$$\text{Therefore } \vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Theorem-2 : ***The necessary and sufficient condition for the coplanarity of the three non-null vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ i.e., $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.***

Proof : The condition is necessary :

Let the three non-null vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ lie in the same plane. Now the vectors $\vec{b}\times\vec{c}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Hence it is perpendicular to $\vec{a}$ [$\because$ $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar].

$\therefore\ \vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ or, $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

The condition is sufficient :

Let $\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\,\vec{b}\,\vec{c}] = 0$

Now $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ implies the vector $\vec{a}$ is perpendicular to $\vec{b}\times\vec{c}$. Again $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Therefore $\vec{a}$ is perpendicular to the normal to the plane of $\vec{b}$ and $\vec{c}$. Hence $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$. Therefore $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Some Observations :

(i) $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}] = -[\vec{b}\,\vec{a}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{c}\,\vec{b}\,\vec{a}]$

As $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot\{-(\vec{c}\times\vec{b})\} = -\vec{a}\cdot(\vec{c}\times\vec{b}) = -[\vec{a}\,\vec{c}\,\vec{b}]$.

Similarly we can show the other relation.

Note

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

and $(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) = -[(\vec{c} . \vec{b})\vec{a} - (\vec{c} . \vec{a})\vec{b}] = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$

$\therefore$ In general $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

$\therefore$ The associative property for vector multiplication is not holds in general.

Particularly, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

or, $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$ or, $(\vec{c} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{b})\vec{c} = \vec{0}$

or, $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{b} \cdot \vec{a})\vec{c} = \vec{0}$ or, $\vec{b} \times (\vec{a} \times \vec{c}) = \vec{0}$

Therefore for three proper vectors $\vec{a}, \vec{b}, \vec{c}$,

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ if $\vec{c}$ is parallel to $\vec{a}$ or $\vec{b}$ is perpendicular to $\vec{a}$ and $\vec{c}$.

Reciprocal System of Vectors

Two system of three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are said to be reciprocal system of vectors *i.e.*, any one system is reciprocal of other if

$$\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1 \quad \text{... (1)}$$

and $\vec{a}\cdot\vec{b}' = \vec{b}\cdot\vec{a}' = \vec{a}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{b}' = 0$ ... (2)

Now we will find the expressions of $\vec{a}', \vec{b}', \vec{c}'$ in terms of $\vec{a}, \vec{b}, \vec{c}$.

From (2) $\vec{a}'\cdot\vec{b} = \vec{a}'\cdot\vec{c} = 0$

$\therefore \vec{a}'$ is perpendicular to $\vec{b}$ and $\vec{c}$

$\therefore \vec{a}'$ is parallel to the vector $\vec{b}\times\vec{c}$.

Let $\vec{a}' = \lambda(\vec{b}\times\vec{c})$ [where λ is a scalar]

Now from (1) $\vec{a}\cdot\vec{a}' = 1$

$\therefore \vec{a}\cdot\lambda(\vec{b}\times\vec{c}) = 1 \Rightarrow \lambda[\vec{a}\ \vec{b}\ \vec{c}] = 1$

$\therefore \lambda = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$ [since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar]

$\therefore \vec{a}' = \dfrac{(\vec{b}\times\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Similarly, we get $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Now, from the symmetry of the equations (1) and (2), we can easily find that

$$\vec{a} = \frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\ \vec{b} = \frac{\vec{c}'\times\vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\ \vec{c} = \frac{\vec{a}'\times\vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$$

Since $[\hat{i}\ \hat{j}\ \hat{k}] = 1$ and $\hat{i}\times\hat{j} = \hat{k}$; $\hat{j}\times\hat{k} = \hat{i}$; $\hat{k}\times\hat{i} = \hat{j}$, we see from the above results that $\hat{i}, \hat{j}, \hat{k}$ is the set of self reciprocal system.

Σ Illustrative Examples

1. **Evaluate the scalar triple product $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]$, where $\vec{\alpha} = \hat{i} + \hat{j} - \hat{k}$, $\vec{\beta} = 3\hat{i} - \hat{k}$, $\vec{\gamma} = 2\hat{i} - 3\hat{j}$. Are $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ coplanar?** [CU 2017]

Solution Here $\vec{\alpha} = \hat{i} + \hat{j} - \hat{k}$, $\vec{\beta} = 3\hat{i} - \hat{k}$, $\vec{\gamma} = 2\hat{i} - 3\hat{j}$

$$\therefore [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = \vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma}) = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 0 & -1 \\ 2 & -3 & 0 \end{vmatrix} = 2(-1) + 3(-1+3) = -2 + 6 = 4$$

Since $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] \neq 0$

$\therefore \vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are not coplanar.

2. **Find the value of the constant d such that the vectors $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} + 2\hat{j} - 3\hat{k})$ and $(3\hat{i} + d\hat{j} + 5\hat{k})$ are coplanar.** [CU 2012, '14]

Solution We know that three vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ will be coplanar if $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$

Here $\vec{\alpha} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{\beta} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{\gamma} = 3\hat{i} + d\hat{j} + 5\hat{k}$

Now, $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$, gives $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & d & 5 \end{vmatrix} = 0$

or, $2(10 + 3d) + 1(5 + 9) + 1(d - 6) = 0$ or, $20 + 6d + 14 + d - 6 = 0$

or, $7d + 28 = 0$ $\therefore d = -\dfrac{28}{7} = -4$

$\therefore$ Required value of d is -4 for which the given three vectors are coplanar.

5. **Prove the identity $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$.** [CH 2010]

<u>Solution</u> $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$

$= (\vec{a}\times\vec{b})\cdot[\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c} - \{(\vec{b}\times\vec{c})\cdot\vec{c}\}\vec{a}]$

$= (\vec{a}\times\vec{b})\cdot[\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c}$ $[\because (\vec{b}\times\vec{c})\cdot\vec{c} = 0]$

$= [(\vec{a}\times\vec{b})\cdot(\vec{b}\ \vec{c}\ \vec{a}]\vec{c} = [\vec{b}\ \vec{c}\ \vec{a}]\,[(\vec{a}\times\vec{b})\cdot\vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]\,[\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]^2$ $[\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]]$

$\therefore [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$ **[Proved]**

6. **Prove that, for any proper vector $\vec{\alpha}$,**
$\hat{i}\times(\vec{\alpha}\times\hat{i}) + \hat{j}\times(\vec{\alpha}\times\hat{j}) + \hat{k}\times(\vec{\alpha}\times\hat{k}) = 2\vec{\alpha}$. [CU 2009, BH 1996]

<u>Solution</u> Let $\vec{\alpha} = \alpha_1\hat{i} + \alpha_2\hat{j} + \alpha_3\hat{k}$

Now, $\hat{i}\times(\vec{\alpha}\times\hat{i}) + \hat{j}\times(\vec{\alpha}\times\hat{j}) + \hat{k}\times(\vec{\alpha}\times\hat{k})$

$= (\hat{i}\cdot\hat{i})\vec{\alpha} - (\hat{i}\cdot\vec{\alpha})\hat{i} + (\hat{j}\cdot\hat{j})\vec{\alpha} - (\hat{j}\cdot\vec{\alpha})\hat{j} + (\hat{k}\cdot\hat{k})\vec{\alpha} - (\hat{k}\cdot\vec{\alpha})\hat{k}$

$= \vec{\alpha} - (\alpha_1)\hat{i} + \vec{\alpha} - (\alpha_2)\hat{j} + \vec{\alpha} - (\alpha_3)\hat{k}$ $[\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\vec{\alpha} = \alpha_1,\ \hat{j}\cdot\vec{\alpha} = \alpha_2,\ \hat{k}.\vec{\alpha} = \alpha_3]$

$= 3\vec{\alpha} - (\alpha_1\hat{i} + \alpha_2\hat{j} + \alpha_3\hat{k}) = 3\vec{\alpha} - \vec{\alpha} = 2\vec{\alpha}$

$\therefore \hat{i}\times(\vec{\alpha}\times\hat{i}) + \hat{j}\times(\vec{\alpha}\times\hat{j}) + \hat{k}\times(\vec{\alpha}\times\hat{k}) = 2\vec{\alpha}.$

17. **Find a set of vectors reciprocal to the set $(\hat{i} + \hat{j} + \hat{k})$, $(2\hat{i} - \hat{j} + 3\hat{k})$ and $(3\hat{i} - 2\hat{j} + \hat{k})$.**

Solution Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{c} = 3\hat{i} - 2\hat{j} + \hat{k}$

Here $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 1(-1 + 6) - 1(2 - 9) + 1(-4 + 3) = 5 + 7 - 1 = 11 \neq 0$

$\therefore\ \vec{a}, \vec{b}, \vec{c}$ are non-coplanar.

Now, if $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ be the set of vectors reciprocal to the given set of vectors, then

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Now, $$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = \hat{i}(-1 + 6) + \hat{j}(9 - 2) + \hat{k}(-4 + 3) = 5\hat{i} + 7\hat{j} - \hat{k}$$

$$\vec{c} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(-2 - 1) - \hat{j}(3 - 1) + \hat{k}(3 + 2) = -3\hat{i} - 2\hat{j} + 5\hat{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \hat{i}(3 + 1) - \hat{j}(3 - 2) + \hat{k}(-1 - 2) = 4\hat{i} - \hat{j} - 3\hat{k}$$

$$\therefore \vec{a}' = \frac{1}{11}(5\hat{i} + 7\hat{j} - \hat{k}),\ \vec{b}' = \frac{1}{11}(-3\hat{i} - 2\hat{j} + 5\hat{k}),\ \vec{c}' = \frac{1}{11}(4\hat{i} - \hat{j} - 3\hat{k})$$

Therefore required set of vectors which is reciprocal to the given set of vectors is

$$\frac{1}{11}(5\hat{i} + 7\hat{j} - \hat{k}),\ \frac{1}{11}(-3\hat{i} - 2\hat{j} + 5\hat{k}) \text{ and } \frac{1}{11}(4\hat{i} - \hat{j} - 3\hat{k}).$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \text{ [using (i)]}$$

Therefore $\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = 0.$ **[Proved]**

31. **If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ be two reciprocal systems of vectors then prove that $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}' = 0$.**

Solution Since $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are two reciprocal system of vectors then we have [from Art. 2.2]

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

31. **If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ be two reciprocal systems of vectors then prove that $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}' = 0$.**

Solution Since $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are two reciprocal system of vectors then we have [from Art. 2.2]

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Now $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}'$

$$= \vec{a} \times \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{b} \times \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{c} \times \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

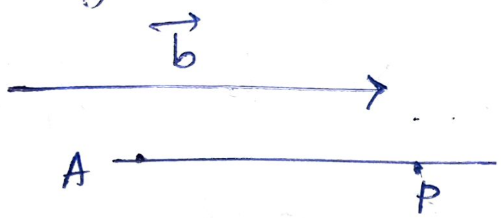
$$= \frac{b(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) + c(\vec{b} \cdot \vec{a}) - \vec{a}(\vec{b} \cdot \vec{c}) + a(\vec{c} \cdot \vec{b}) - \vec{b}(\vec{c} \cdot \vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]} = O$$

32 If th

Vector equation of a straight line

(1) Equation of a St. line parallel to a given vector $(\vec{b})$ and passing through given point $(\vec{a})$ is

$$\vec{r} = \vec{a} + t\vec{b}$$



(2) Equ^n of a st. line through given two points $\vec{a}$ and $\vec{b}$ is

$$\vec{r} = (1-t)\,\vec{a} + t\vec{b}$$

(3) Vector equ^n of a St. line in non-parametric form is $(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$ which parallel to $\vec{b}$ and passes through $\vec{a}$.

(4) perpendicular distance of a point $(\vec{p})$ from a St. line $(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$ is

$$|(\vec{p} - \vec{a}) \times \vec{b}|$$

(5) Shortest distance between two Skew lines $\vec{r} = \vec{a} + t\vec{\alpha}$ and $\vec{r} = \vec{b} + s\vec{\beta}$ is

$$\left|(\vec{b} - \vec{a}) \cdot \frac{\vec{\alpha} \times \vec{\beta}}{|\vec{\alpha} \times \vec{\beta}|}\right|$$

Vector equn of a plane

① vector equn of a plane through a given point($\vec{\alpha}$) and parallel to two vectors $\vec{\beta}$ and $\vec{\gamma}$ is

$$\vec{r} = \vec{\alpha} + t\vec{\beta} + s\vec{\gamma}$$

② Vector equn of a plane through two points $\vec{\alpha}$ and $\vec{\beta}$, parallel to $\vec{\gamma}$

is

$$\vec{r} = (1-s)\vec{\alpha} + s\vec{\beta} + t\vec{\gamma}$$

③ vector equn of a plane which is perpendicular to a unit vector ~~and~~ and passing through a given point is

$$\vec{r}\cdot\hat{n} = p \quad \text{where } p = \vec{\alpha}\cdot\hat{n}$$

④ perpendicular distance of a point ~~P~~ ($\vec{P}$) from a plane $\vec{r}\cdot\vec{n} = p$ is

$$\frac{|p - \vec{a}\cdot\vec{n}|}{|\vec{n}|}, \text{ where } \vec{n} \text{ is in}$$

the direction of the normal to the plane

- ## Work done by Force

Due to the force $\vec{F}$ if the point of application of this force experiences a displacement represented by the vector $\vec{d}$. Then the work done is $\vec{F} \cdot \vec{d}$

- Moment of a force $\vec{F}$ ~~about a~~ applied to A of a body and O be any point is (Let $\vec{OA} = \vec{r}$) $|\vec{r} \times \vec{F}|$

- Moment of a force about a line whose direction is $\vec{r}$ is $\vec{m} = \vec{r} \times \vec{F}$

2. **Find the vector equation of a straight line passing through the origin and parallel to the vector $(\hat{i} - 2\hat{j} + 3\hat{k})$.**

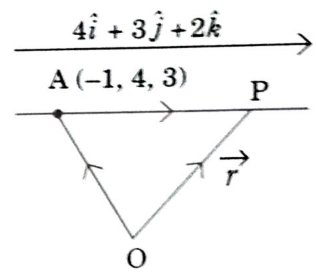
Solution Let P be any point on the line whose positive vector is $\vec{r}$.

Since we have the equation of the straight line passing through the origin and parallel to the vector $\vec{b}$ is $\vec{r} = t\vec{b}$, where t is a scalar.

So the required equation of the straight line is $\vec{r} = t(\hat{i} - 2\hat{j} + 3\hat{k})$.

3. **Find the vector equation of the straight line passing through the point (–1, 4, 3) and parallel to the vector $4\hat{i} + 3\hat{j} + 2\hat{k}$.** [CU 2016, 2014]

Solution Let A be the point (–1, 4, 3) [See Fig.]



$\therefore \overrightarrow{OA} = -\hat{i} + 4\hat{j} + 3\hat{k}$ [where O is the origin]

Let P be any point on the line, whose position vector is $\vec{r}$.

Now $\overrightarrow{AP} = \vec{r} - (-\hat{i} + 4\hat{j} + 3\hat{k})$

Now $\overrightarrow{AP}$ is parallel to the vector $4\hat{i} + 3\hat{j} + 2\hat{k}$.

$\therefore$ Required equation at the straight line in parametric form is

$\vec{r} - (-\hat{i} + 4\hat{j} + 3\hat{k}) = t(4\hat{i} + 3\hat{j} + 2\hat{k})$ [t is a scalar]

or, $\vec{r} = (-\hat{i} + 4\hat{j} + 3\hat{k}) + t(4\hat{i} + 3\hat{j} + 2\hat{k}) = (4t - 1)\hat{i} + (3t + 4)\hat{j} + (2t + 3)\hat{k}$.

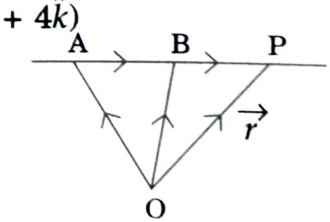
4. **Find the vector equation of a straight line passing through the points (2, 3, 4) and (–1, 3, – 2).** [CU 2011, '13, '15]

Solution Let A and B be the two points whose coordinates are (2, 3, 4) and (–1, 3, – 2).

Hence $\overrightarrow{OA} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overrightarrow{OB} = -\hat{i} + 3\hat{j} - 2\hat{k}$, where O is the origin [See Fig.]

Let P be any point on the line whose position vector is $\vec{r}$.

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -3\hat{i} - 6\hat{k}$ and $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - (2\hat{i} + 3\hat{j} + 4\hat{k})$



Now $\overrightarrow{AP}$ is parallel to $\overrightarrow{AB}$.

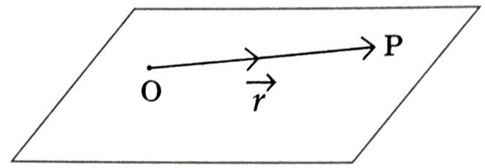
$\therefore$ Required equation of the straight line in parametric form is

$\vec{r} - (2\hat{i} + 3\hat{j} + 4\hat{k}) = t(-3\hat{i} - 6\hat{k})$, where t is a scalar

or, $\vec{r} = (2 - 3t)\hat{i} + 3\hat{j} + (4 - 6t)\hat{k}$.

10. **Find the vector equation of a plane passing through the origin and parallel to the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $2\hat{i} - \hat{j} - 4\hat{k}$.** [CU 2013, 2015]

Solution The plane passes through the origin O and let P be any point on the plane whose position vector is $\vec{r}$. Again the plane is parallel to the vectors $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - 4\hat{k}$.



$\therefore \vec{r}, \vec{a}$ and $\vec{b}$ are coplanar.

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= \hat{i}(-8+3) + \hat{j}(6+4) + \hat{k}(-1-4) = -5\hat{i} + 10\hat{j} - 5\hat{k}$$

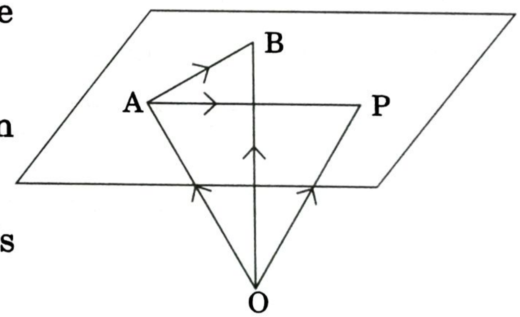
So the required equation of the plane is $[\vec{r}\ \vec{a}\ \vec{b}] = \vec{r} \cdot (\vec{a} \times \vec{b}) = 0$

i.e., $\vec{r} \cdot (-5\hat{i} + 10\hat{j} - 5\hat{k}) = 0$.

11. **Find the vector equation of the plane passing through the points (1, 1, 1) and (2, – 1, 3) and parallel to the vector $3\hat{i} - 4\hat{j} + 4\hat{k}$.**

Solution Let A(1, 1, 1) and B(2, –1, 3) be the points on the plane, then

$\vec{OA} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{OB} = 2\hat{i} - \hat{j} + 3\hat{k}$, where O is the Origin (See Fig.)



Let P be any point on the plane whose position vector is $\vec{r}$. The plane is parallel to the vector $\vec{\gamma} = 3\hat{i} - 4\hat{j} + 4\hat{k}$.

Now $\vec{AP}$, $\vec{AB}$ and $\vec{\gamma}$ are coplanar. So $\vec{AP}$ can be written as a linear combination of $\vec{AB}$ and $\vec{\gamma}$ as $\vec{AP} = s\vec{AB} + t\vec{\gamma}$ ... (1)

Now $\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - (\hat{i} + \hat{j} + \hat{k})$

$\vec{AB} = \vec{OB} - \vec{OA} = (2\hat{i} - \hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} - 2\hat{j} + 2\hat{k}$.

Then (1) becomes, $\vec{r} - (\hat{i} + \hat{j} + \hat{k}) = s(\hat{i} - 2\hat{j} + 2\hat{k}) + t(3\hat{i} - 4\hat{j} + 4\hat{k})$

This is the required equation of the plane.

18. **Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot \vec{a} + 9 = 0$, where $\vec{a} = 5\hat{i} + 2\hat{j} - 7\hat{k}$. Also show that the points lie on opposite sides of the plane.** [CH 2006]

Solution Here the equation of the plane is $\vec{r} \cdot \vec{a} + 9 = 0$ ... (1)

where $\vec{a} = 5\hat{i} + 2\hat{j} - 7\hat{k}$

So $|\vec{a}| = \sqrt{5^2 + 2^2 + (-7)^2} = \sqrt{25 + 4 + 49} = \sqrt{78}$

Now (1) can be written as

$$\vec{r} \cdot \frac{\vec{a}}{|\vec{a}|} = -\frac{9}{|\vec{a}|}$$

or, $\vec{r} \cdot \frac{\vec{a}}{\sqrt{78}} = -\frac{9}{\sqrt{78}}$ or, $p = \frac{9}{\sqrt{78}} = \vec{r} \cdot \left(\frac{-5\hat{i} - 2\hat{j} + 7\hat{k}}{\sqrt{78}}\right)$

Therefore $\hat{n} = \frac{1}{\sqrt{78}}(-5\hat{i} - 2\hat{j} + 7\hat{k})$

The distance of the point $(\hat{i} - \hat{j} + 3\hat{k})$ from the plane is

$$\left[\frac{9}{\sqrt{78}} - (\hat{i} - \hat{j} + 3\hat{k}) \cdot \left(\frac{-5\hat{i} - 2\hat{j} + 7\hat{k}}{\sqrt{78}}\right)\right] = \frac{9}{\sqrt{78}} - \frac{(-5+2+21)}{\sqrt{78}} = -\frac{9}{\sqrt{78}}$$

The distance of the point $3(\hat{i} + \hat{j} + 3\hat{k})$ from the plane is

$$\frac{9}{\sqrt{78}} - \frac{3(\hat{i} + \hat{j} + \hat{k}) \cdot (-5\hat{i} - 2\hat{j} + 7\hat{k})}{\sqrt{78}} = \frac{9}{\sqrt{78}} - \frac{3(-5-2+7)}{\sqrt{78}} = \frac{9}{\sqrt{78}} \text{ units}$$

Since the magnitudes of these two is same so the given two points are equidistant from the plane. The signs of these two are opposite, so the two points lie on opposite sides of the plane.

19. **Using vector method find the distance of the point (2, 3, 4) to the plane $3x - 6y + 2z + 11 = 0$ measure along the vector (3, 1, 2).** [CH 2010]

Solution The equation of the plane in normal form is $\vec{r} \cdot \vec{n} = p$ ... (1)

where $\vec{n} = -3\hat{i} + 6\hat{j} - 2\hat{k}$, $p = 11$

Position vector of the point from where the distance is measured is $\vec{\alpha} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

The distance is measured along the vector $3\hat{i} + \hat{j} + 2\hat{k}$.

This unit vector along this direction $\vec{\beta} = \frac{1}{\sqrt{14}}(3\hat{i} + \hat{j} + 2\hat{k})$

Now the required distance is

$$= \left|\frac{p - \vec{\alpha} \cdot \vec{n}}{\vec{\beta} \cdot \vec{n}}\right| = \left|\frac{11 - (2\hat{i}+3\hat{j}+4\hat{k}) \cdot (-3\hat{i}+6\hat{j}-2\hat{k})}{\frac{1}{\sqrt{14}}(3\hat{i}+\hat{j}+2\hat{k}) \cdot (-3\hat{i}+6\hat{j}-2\hat{k})}\right| = \left|\frac{11-(-6+18-8)}{\frac{1}{\sqrt{14}}(-9+6-4)}\right| = \left|\frac{11-4}{\frac{1}{\sqrt{14}}(-7)}\right| = \sqrt{14} \text{ units.}$$

27. **Find the work done by the force $4\hat{i} + 5\hat{j} + 3\hat{k}$, which displaces a particle from the point (–1, – 1, 3) to the point (1, 1, 2).** [CU 2012]

Solution Here the force $\vec{F} = 4\hat{i} + 5\hat{j} + 3\hat{k}$ displaces the particle from the point $-\hat{i} - \hat{j} + 3\hat{k}$ to the point $\hat{i} + \hat{j} + 2\hat{k}$.

So the displacement vector $\vec{d} = (\hat{i} + \hat{j} + 2\hat{k}) - (-\hat{i} - \hat{j} + 3\hat{k}) = 2\hat{i} + 2\hat{j} - \hat{k}$

Therefore the work done by the force

$= \vec{F} \cdot \vec{d} = (4\hat{i} + 5\hat{j} + 3\hat{k}) \cdot (2\hat{i} + 2\hat{j} - \hat{k})$

$= 4 \cdot 2 + 5 \cdot 2 + 3(-1) = 8 + 10 - 3 = 15$ units of work.

28. **A particle acted on the constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ is displaced from the point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. Find the amount of work done.** [CH 1998, 2011]

Solution Here the given two forces are $\vec{F_1} = 4\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{F_2} = 3\hat{i} + \hat{j} - \hat{k}$.

Now the resultant of the two forces is $\vec{F} = \vec{F_1} + \vec{F_2} = (4\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + \hat{j} - \hat{k})$

$= 7\hat{i} + 2\hat{j} - 4\hat{k}$.

Here the forces displace the particle from the point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$.

So the displacement vector $\vec{d} = (5\hat{i} + 4\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} + 2\hat{j} - 2\hat{k}$

Therefore the total work done by the forces

$= \vec{F} \cdot \vec{d} = (7\hat{i} + 2\hat{j} - 4\hat{k}).\ (4\hat{i} + 2\hat{j} - 2\hat{k}) = 28 + 4 + 8 = 40$ units of work.

32. **A force $5\hat{i} + 2\hat{j} - 3\hat{k}$ is applied at the point (1, –2, 2). Find the value of the moment of the force about the origin.** [CU 2014]

Solution Here the position vector of the point of application of the force with respect to the origin is $\vec{r} = \hat{i} - 2\hat{j} + 2\hat{k}$ and the force is $\vec{F} = 5\hat{i} + 2\hat{j} - 3\hat{k}$

Hence the required moment is $\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 5 & 2 & -3 \end{vmatrix}$

$$= \hat{i}(6 - 4) - \hat{j}(-3 - 10) + \hat{k}(2 + 10) = 2\hat{i} + 13\hat{j} + 12\hat{k}$$

33. **A force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ is applied at the point (1, – 1, 2). Find the moment of $\vec{F}$ about the point (2, – 1, 3). Find its magnitude.** [CU 2016]

Solution Here the position vector of the point of application of the force is $\hat{i} - \hat{j} + 2\hat{k}$ and we have to find the moment of the force about the point (2, – 1, 3).

So here $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) = -\hat{i} - \hat{k}$

Here the force is $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

So the required moment $= \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = \hat{i}(0 + 2) - \hat{j}(4 + 3) + \hat{k}(-2) = 2\hat{i} - 7\hat{j} - 2\hat{k}$$

And its magnitude is $|2\hat{i} - 7\hat{j} - 2\hat{k}| = \sqrt{2^2 + (-7)^2 + (-2)^2} = \sqrt{4 + 49 + 4} = \sqrt{57}$

4.2 Limit of a Vector Valued Function

A vector valued function $\vec{F}(t)$ is said to tend to a limit $\vec{A}$ (a constant vector) as t approaches to t_0, if to any pressigned number ε (> 0) however small, there corresponds a number δ(> 0) such that

$$\left|\vec{F}(t)-\vec{A}\right| < \varepsilon, \text{ whenever } 0 < |t-t_0| < \delta.$$

Symbolically, we express this fact by $\lim\limits_{t \to t_0} \vec{F}(t) = \vec{A}$.

Results :

① If $\vec{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$ and $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$,
($\hat{i}$, $\hat{j}$, $\hat{k}$: three unit vector along three fixed mutually perpendicular directions)
then $\lim\limits_{t \to t_0} \vec{F}(t) = \vec{A} \Leftrightarrow$ (i) $\lim\limits_{t \to t_0} F_1(t) = A_1$, (ii) $\lim\limits_{t \to t_0} F_2(t) = A_2$ and (iii) $\lim\limits_{t \to t_0} F_3(t) = A_3$.

② If $\vec{F}(t)$ and $\vec{G}(t)$ are two vector valued functions of a scalar variable t and $\phi(t)$ be a scalar function of t, then following relations hold.

(i) $\lim\limits_{t \to t_0} \left[\vec{F}(t) \pm \vec{G}(t)\right] = \lim\limits_{t \to t_0} \vec{F}(t) \pm \lim\limits_{t \to t_0} \vec{G}(t)$

(ii) $\lim\limits_{t \to t_0} \left[\vec{F}(t) \cdot \vec{G}(t)\right] = \left[\lim\limits_{t \to t_0} \vec{F}(t)\right] \cdot \left[\lim\limits_{t \to t_0} \vec{G}(t)\right]$

(iii) $\lim\limits_{t \to t_0} \left[\vec{F}(t) \times \vec{G}(t)\right] = \left[\lim\limits_{t \to t_0} \vec{F}(t)\right] \times \left[\lim\limits_{t \to t_0} \vec{G}(t)\right]$

(iv) $\lim\limits_{t \to t_0} \left[\phi(t)\vec{F}(t)\right] = \left[\lim\limits_{t \to t_0} \phi(t)\right]\left[\lim\limits_{t \to t_0} \vec{F}(t)\right]$ (all the limits are assumed to exist)

4.6 Derivatives of Sums and Products of Vector Valued Functions

If $\vec{u}(t), \vec{v}(t), \vec{w}(t)$ are three differentiable vector valued functions of a scalar variable t and $\phi(t)$ is a differentiable scalar function of t, then

(i) $\dfrac{d}{dt}(\vec{u}+\vec{v}) = \dfrac{d\vec{u}}{dt}+\dfrac{d\vec{v}}{dt}$,

(ii) $\dfrac{d}{dt}(\phi\,\vec{u}) = \phi\dfrac{d\vec{u}}{dt}+\dfrac{d\phi}{dt}\vec{u}$,

(iii) $\dfrac{d}{dt}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\dfrac{d\vec{v}}{dt}+\dfrac{d\vec{u}}{dt}\cdot\vec{v}$,

(iv) $\dfrac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\dfrac{d\vec{v}}{dt}+\dfrac{d\vec{u}}{dt}\times\vec{v}$,

(v) $\dfrac{d}{dt}\left[\vec{u}\ \vec{v}\ \vec{w}\right] = \left[\dfrac{d\vec{u}}{dt}\ \vec{v}\ \vec{w}\right]+\left[\vec{u}\ \dfrac{d\vec{v}}{dt}\ \vec{w}\right]+\left[\vec{u}\ \vec{v}\ \dfrac{d\vec{w}}{dt}\right]$,

4.7 Three Theorems concerning $\vec{F}(t)$ to be Constant, Constant in Magnitude and Constant in Direction

Theorem-3 : The necessary and sufficient condition for the vector valued function $\vec{F}(t)$ to be constant is $\frac{d\vec{F}}{dt} = \vec{0}$.

Proof :

Necessary Part : Given : $\vec{F}(t)$ be a constant vector valued function of a scalar variable t.

To be proved : $\frac{d\vec{F}}{dt} = \vec{0}$.

As $\vec{F}(t)$ is constant, for an increment Δt in t, $\vec{F}(t)$ will be unchanged. Then $\Delta\vec{F} = \vec{0}$.

Hence $\frac{\Delta\vec{F}}{\Delta t} = \vec{0}$

$\Rightarrow \frac{d\vec{F}}{dt} = \vec{0}$ (taking limits both sides as $\Delta t \to 0$)

Sufficient Part : Given : $\frac{d\vec{F}}{dt} = \vec{0}$.

To be proved : $\vec{F}(t)$ be a constant vector valued function.

We consider $\vec{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$

$\Rightarrow \frac{d\vec{F}}{dt} = \frac{dF_1}{dt}\hat{i} + \frac{dF_2}{dt}\hat{j} + \frac{dF_3}{dt}\hat{k} = \vec{0}$ [given]

$\Rightarrow \frac{dF_1}{dt} = 0, \frac{dF_2}{dt} = 0, \frac{dF_3}{dt} = 0$

$\Rightarrow$ each of F_1, F_2 and F_3 is constant.

$\Rightarrow \vec{F}(t)$ be a constant vector valued function.

Theorem-4 : A necessary and sufficient condition that a non-zero vector valued function $\vec{F}(t)$ to be of constant magnitude is that $\vec{F}\cdot\frac{d\vec{F}}{dt} = 0$

Proof :

Necessary Part : Given : A non-zero vector valued function $\vec{F}(t)$ has a constant magnitude, *i.e.*, constant length, *i.e.*, $|\vec{F}(t)|$ is constant.

To be proved : $\vec{F}\cdot\frac{d\vec{F}}{dt} = 0$.

We know that $\vec{F}\cdot\vec{F} = \vec{F}^2 = |\vec{F}|^2$

$\Rightarrow \frac{d}{dt}(\vec{F}\cdot\vec{F}) = 2\vec{F}\cdot\frac{d\vec{F}}{dt} = \frac{d}{dt}\{|\vec{F}|^2\} = 0$ [since $|\vec{F}|$ is constant]

$\Rightarrow \vec{F}\cdot\frac{d\vec{F}}{dt} = 0$

Sufficient Part : **Given :** $\vec{F}\cdot\frac{d\vec{F}}{dt}=0$

To be proved : $|\vec{F}|$ is constant, *i.e.* $\vec{F}$ has a constant magnitude.

Now $0=\vec{F}\cdot\frac{d\vec{F}}{dt}=2\vec{F}\cdot\frac{d\vec{F}}{dt}=\frac{d}{dt}(\vec{F}\cdot\vec{F})=\frac{d}{dt}\{|\vec{F}|^2\}$

$\Rightarrow |\vec{F}|$ is constant, *i.e.*, $\vec{F}$ has a constant magnitude.

Theorem-5 : A necessary and sufficient condition that a non-zero vector valued function $\vec{F}(t)$ to be of constant direction is that $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$.

Proof :

Necessary Part : **Given :** A non-zero vector valued function $\vec{F}(t)$ has a constant direction, *i.e.*, $\vec{F}(t)$ remains parallel to a fixed line always.

To be proved : $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$.

Let $\vec{F}(t)=|\vec{F}(t)|\,\hat{F}(t)$

where $\hat{F}(t)$ is the unit vector in the direction of $\vec{F}(t)$ and $|\vec{F}(t)|$ be the length of $\vec{F}(t)$.

Now $\vec{F}\times\frac{d\vec{F}}{dt}=\left(|\vec{F}|\hat{F})\times\frac{d}{dt}(|F|\hat{F}\right)=|\vec{F}|\hat{F}\times\left(|\vec{F}|\frac{d\hat{F}}{dt}+\frac{d|\vec{F}|}{dt}\hat{F}\right)$

$=|\vec{F}|^2\,\hat{F}\times\frac{d\hat{F}}{dt}+|\vec{F}|\frac{d|\vec{F}|}{dt}(\hat{F}\times\hat{F})=|\vec{F}|^2\,\hat{F}\times\frac{d\hat{F}}{dt}$ (since $\hat{F}\times\hat{F}=\vec{0}$) ... (1)

Again,

(i) $\hat{F}$ has a constant direction (since $\vec{F}$ has a constant direction and $\hat{F}$ and $\vec{F}$ have same direction) and

(ii) $\hat{F}$ has a constant magnitude (since $\hat{F}$ is a unit vector).

Hence $\hat{F}$ has both constant magnitude and direction.

Therefore, $\hat{F}$ is a constant vector valued function.

$\Rightarrow \frac{d\hat{F}}{dt}=\vec{0}$ ... (2)

Thus, from (1) and using (2), we obtain $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$.

Sufficient Part : **Given :** $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$.

To be proved : $|\vec{F}|$ has a constant direction.

We have $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$

$\Rightarrow |\vec{F}|^2\,\hat{F}\times\frac{d\hat{F}}{dt}=\vec{0}$ [from (1)]

$\Rightarrow \hat{F}\times\frac{d\hat{F}}{dt}=\vec{0}$. [$|\vec{F}|\neq 0$, as $\vec{F}$ is non-zero] ... (3)

Again as $\hat{F}$ has a constant magnitude ($|\hat{F}| = 1$), we also have

$$\hat{F} \cdot \frac{d\hat{F}}{dt} = 0 \cdot \qquad \text{... (4)}$$

Now (3) and (4) hold simultaneously, if $\frac{d\hat{F}}{dt} = 0$

i.e., if $\hat{F}$ is a constant vector valued function

i.e., if $\hat{F}$ has a constant direction (since $\hat{F}$ is of constant magnitude already)

i.e., if $\vec{F}$ has a constant direction (since $\hat{F}$ and $\vec{F}$ have same direction)

② $\int\left(\vec{r}\cdot\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{s}\right)dt = \vec{r}\cdot\vec{s}+c.$

$\left[\text{since}\ \frac{d}{dt}(\vec{r}\cdot\vec{s})=\vec{r}\cdot\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{s} \Rightarrow d(\vec{r}\,.\,\vec{s}\,)=\left(\vec{r}\cdot\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{s}\right)dt\right]$

③ $\int\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \vec{r}\,.\,\vec{r}+c = \vec{r}^{\,2}+c.$

[in **formula 2**, putting $\vec{r}=\vec{s}$,

$\int\left(\vec{r}\cdot\frac{d\vec{r}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{r}\right)dt = \vec{r}\cdot\vec{r}+c \Rightarrow \int\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \vec{r}\cdot\vec{r}+c=\vec{r}^{\,2}+c$]

④ $\int\left(2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right)dt = \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}+c=\left(\frac{d\vec{r}}{dt}\right)^2+c$

[by replacing $\vec{r}$ by $\frac{d\vec{r}}{dt}$ in **formula 3**]

⑤ $\int\left(\vec{r}\times\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\times\vec{s}\right)dt = (\vec{r}\times\vec{s}\,)+\vec{c}\,.$

$\left[\text{we have}\ \frac{d}{dt}(\vec{r}\times\vec{s}\,)=\vec{r}\times\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\times\vec{s} \Rightarrow d(\vec{r}\times\vec{s}\,)=\left(\vec{r}\times\frac{d\vec{s}}{dt}+\frac{d\vec{r}}{dt}\times\vec{s}\right)dt\right]$

⑥ $\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)+\vec{c}\,.$

$\left[\text{we have}\ \frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\vec{r}\times\frac{d^2\vec{r}}{dt^2} \Rightarrow d\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt\right]$

⑦ $\int\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{dr}{dt}\frac{\vec{r}}{r^2}\right)dt=\frac{\vec{r}}{r}+\vec{c}$. (r is a scalar function of t)

$\left[\text{we have}\ \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)=\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r} \Rightarrow d\left(\frac{\vec{r}}{r}\right)=\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{dr}{dt}\frac{\vec{r}}{r^2}\right)dt\right]$

⑧ $\int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = (\vec{a}\times\vec{r}\,)+\vec{c}\,.$ [$\vec{a}$ is a constant vector]

[we have $\frac{d}{dt}(\vec{a}\times\vec{r}\,)=\vec{a}\times\frac{d\vec{r}}{dt}$ (since $\vec{a}$ is a constant vector)

$\Rightarrow d(\vec{a}\times\vec{r}\,)=\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt$]

1. **If $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{s} = \sin t\,\hat{i} - \cos t\,\hat{j}$, find the values of (i) $\frac{d}{dt}(\vec{r}\cdot\vec{s})$ at $t = 0$ (ii) $\frac{d}{dt}(\vec{r}\times\vec{s})$ at $t = 0$ (iii) $\frac{d}{dt}(\vec{r}\cdot\vec{r})$ at $t = 1$.**

Solution Given : $\vec{r} = (5t^2, t, -t^3)$ and $\vec{s} = (\sin t, -\cos t, 0)$

(i) $\vec{r}\cdot\vec{s} = (5t^2, t, -t^3)\cdot(\sin t, -\cos t, 0) = 5t^2\sin t - t\cos t$

Hence $\frac{d}{dt}(\vec{r}\cdot\vec{s}) = 5(t^2\cos t + 2t\sin t) - (\cos t - t\sin t) = (5t^2 - 1)\cos t + 11t\sin t.$

Therefore, $\left.\frac{d}{dt}(\vec{r}\cdot\vec{s})\right|_{t=0} = -1.$

(ii) $\vec{r}\times\vec{s} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = -t^3\cos t\,\hat{i} - t^3\sin t\,\hat{j} - (5t^2\cos t + t\sin t)\,\hat{k}.$

Hence $\frac{d}{dt}(\vec{r}\times\vec{s}) = (t^3\sin t - 3t^2\cos t)\,\hat{i} - (t^3\cos t + 3t^2\sin t)\,\hat{j} + (5t^2\sin t - \sin t - 11t\cos t)\hat{k}.$

Therefore, $\left.\frac{d}{dt}(\vec{r}\times\vec{s})\right|_{t=0} = \vec{0}.$

(iii) $\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\frac{d\vec{r}}{dt} = 2(5t^2, t, -t^3)\cdot\frac{d}{dt}(5t^2, t, -t^3)$

$= 2(5t^2, t, -t^3)\cdot(10t, 1, -3t^2) = 2(50t^3 + t + 3t^5)$

Therefore, $\left.\frac{d}{dt}(\vec{r}\cdot\vec{r})\right|_{t=1} = 108.$

2. **If $\vec{\alpha} = t^2\hat{i} - t\hat{j} + (2t + 1)\hat{k}$ and $\vec{\beta} = (2t - 3)\hat{i} + \hat{j} - t\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ have their usual meanings, then find $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right)$ at $t = 2$.** **[CH 1981]**

Solution $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right) = \vec{\alpha}\times\frac{d^2\vec{\beta}}{dt^2} + \frac{d\vec{\alpha}}{dt}\times\frac{d\vec{\beta}}{dt}$ ... (1)

Given that $\vec{\alpha} = t^2\hat{i} - t\hat{j} + (2t + 1)\hat{k} \Rightarrow \frac{d\vec{\alpha}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k}.$

Again, $\vec{\beta} = (2t - 3)\hat{i} + \hat{j} - t\hat{k} \Rightarrow \frac{d\vec{\beta}}{dt} = 2\hat{i} - \hat{k}, \frac{d^2\beta}{dt^2} = \vec{0}.$

Thus from (1), $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \hat{i} + (2t + 4)\hat{j} + 2\hat{k}$

Hence $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right)$ at $t = 2$ is $\hat{i} + 8\hat{j} + 2\hat{k}.$

8. **If $\vec{r} = (3t, 3t^2, 2t^3)$, find $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$.** [CH 2002]

Solution **Given :** $\vec{r} = (3t, 3t^2, 2t^3)$

$$\Rightarrow \frac{d\vec{r}}{dt} = (3, 6t, 6t^2),\quad \frac{d^2\vec{r}}{dt^2} = (0, 6, 12\,t) \text{ and } \frac{d^3\vec{r}}{dt^3} = (0, 0, 12).$$

Hence $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} 3 & 6t & 6t^2 \\ 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix} = 3 \times 72 = 216.$

17. **Evaluate** $\int_2^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$, **where** $\vec{r} = t^3\hat{i} + 2t^2\hat{j} + 3t\hat{k}$. [CH 2019]

Solution We have $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \left(\vec{r} \times \frac{d\vec{r}}{dt}\right) + \vec{c}$

Therefore, $\int_2^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \left[\left(t^3\hat{i} + 2t^2\hat{j} + 3t\hat{k}\right) \times \left(3t^2\hat{i} + 4t\hat{j} + 3\hat{k}\right)\right]_{t=2}^3$

$$= \left[-6t^2\hat{i} + 6t^3\hat{j} - 2t^4\hat{k}\right]_{t=2}^3$$

$$= (-54\hat{i} + 162\hat{j} - 162\hat{k}) - (-24\hat{i} + 48\hat{j} - 32\hat{k})$$

$$= -30\hat{i} + 114\hat{j} - 130\hat{k}.$$

7. **If $\vec{r} = (a\cos t, a\sin t, at\tan\alpha)$, then evaluate**

(i) $\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right|$ **and (ii)** $\dfrac{d\vec{r}}{dt} \cdot \dfrac{d^2\vec{r}}{dt^2} \times \dfrac{d^3\vec{r}}{dt^3}$. [CH 2009, 2012, 2016]

Solution Given that $\vec{r} = (a\cos t, a\sin t, at\tan\alpha) \Rightarrow \dfrac{d\vec{r}}{dt} = (-a\sin t, a\cos t, a\tan\alpha)$, ... (1)

$\dfrac{d^2\vec{r}}{dt^2} = (-a\cos t, -a\sin t, 0)$ ... (2) and $\dfrac{d^3\vec{r}}{dt^3} = (a\sin t, -a\cos t, 0)$ ... (3)

Thus, **(i)** $\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$ [by (1) and (2)]

$$= (a^2\tan\alpha\sin t, -a^2\tan\alpha\cos t, a^2)$$

Therefore, $\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4(\tan^2\alpha\sin^2 t + \tan^2\alpha\cos^2 t + 1)}$

$$= \sqrt{a^4(\tan^2\alpha + 1)} = \sqrt{a^4\sec^2\alpha} = a^2\sec\alpha.$$

(ii) $\dfrac{d\vec{r}}{dt} \cdot \dfrac{d^2\vec{r}}{dt^2} \times \dfrac{d^3\vec{r}}{dt^3} = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$ [by (1), (2) and (3)]

$$= a\tan\alpha\ (a^2\cos^2 t + a^2\sin^2 t) = a\tan\alpha \cdot a^2 = a^3\tan\alpha.$$