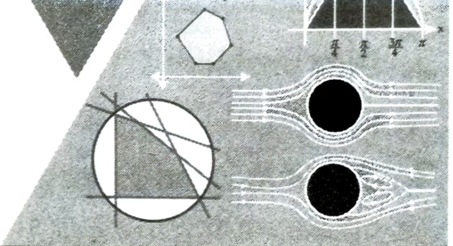
Formulation of a Linear Programming Problem



1.1 What is Linear Programming

Linear programming is a mathematical procedure for determining the optimum (maximum or minimum) value of a linear multivariable function (called objective function) subject to some constraints (linear equations and/or inequations). The non-negativity condition of variables (*i.e.*, feasibility) is also a constraint.

3. **Food X contains 6 units of vitamin A and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A and 12 units of vitamin B per gram and costs 20 paise per gram. The daily requirements of vitamins A and B are at least 100 units and 120 units respectively. Formulate this problem as a LPP defining variables suitably.**

[CP 2016, 2019]

Solution Here the key decision to be made is to purchase the quantity of two food products X and Y to minimize the cost, while satisfying the given requirements of vitamins.

Here the variables are

x_1 = Amount of food X in gms to be purchased.

x_2 = Amount of food Y in gms to be purchased

Since it is not possible to purchase negative quantity of food, so $x_1 \geq 0$, $x_2 \geq 0$.

The amount of vitamin A present in two type of food is $6x_1 + 8x_2$ and that of vitamin B is $7x_1 + 12x_2$. Now the minimum requirements of vitamin A and B are 100 units and 120 units respectively. So $6x_1 + 8x_2 \geq 100$; $7x_1 + 12x_2 \geq 120$.

In this problem the objective is to minimize the cost of purchasing the two types of food, which is given by Minimize (Total cost) $z = 12x_1 + 20x_2$.

Hence the given problem can be expressed in Linear Programming format as follows :

Minimize $z = 12x_1 + 20x_2$

Subject to $6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

$x_1 \geq 0, x_2 \geq 0.$

8. **Three different types of lorries A, B and C have been used to transport 80 tons solid and 45 tons liquid substances. Each A type lorry can carry 9 tons solid and 5 tons liquid. Each B type lorry can carry 8 tons solid and 3 tons liquid and each C type lorry can carry 5 ton solid and 6 tons liquid. The costs of transport are ₹ 600, ₹ 500 and ₹ 400 per lorry of A, B and C type respectively. Formulate the above problem as a linear programming problem to minimize the transportation cost.**

Solution Here the key decision to be made is to determine the number of three type lorries to be used to minimize the total transportation cost to transport given amount of solid and liquid substances.

Here the variables are identified as :

x_1 = The number of A type lorries, which are used to transport the materials.

x_2 = The number of B type lorries, which are used to transport the materials

and x_3 = The number of C type lorries, which are used to transport the materials.

The number of lorries cannot be negative.

So $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$

The quantity of solid substances in tons transported by the lorries is $9x_1 + 8x_2 + 5x_3$.

The quantity of liquid substances in tons, transported by the lorries is $5x_1 + 3x_2 + 6x_3$.

By the given conditions $9x_1 + 8x_2 + 5x_3 \geq 80$ and $5x_1 + 3x_2 + 6x_3 \geq 45$.

In this problem the objective is to minimize the transportation cost, which is given by

Minimize (Total cost) $z = 600x_1 + 500x_2 + 400x_3$

Thus the required linear programming is

$$
\begin{aligned}
\text{Min. } z &= 600x_1 + 500x_2 + 400x_3 \\
\text{Subject to } & 9x_1 + 8x_2 + 5x_3 \geq 80 \\
& 5x_1 + 3x_2 + 6x_3 \geq 45 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

2.14 Basic Solution

We consider a system of m simultaneous linear equations in n unknowns $AX = \boldsymbol{b}$; $X \in E^n$, where A is a $m \times n$ matrix of rank $m(m < n)$. Let B be any $m \times m$ non-singular submatrix of A. Then a solution obtained by setting $(n - m)$ variables not associated with the columns of B, equal to zero, and solving the resulting system, is called a Basic solution to the given system of equations.

The m variables, which may be different from zero, are called Basic Variables. The $m \times m$ non-zingular submatrix B is called Basis Matrix and the columns of B as Basis vectors.

2.16 Feasible and Basic Feasible Solution

■ Feasible solution

A solution of a linear programming problem which satisfies the constraints and non-negativity conditions is called feasible solution (F.S.).

■ Basic feasible solution

In linear programming problem a solution, which is feasible and as well as basic, is called a basic feasible solution (B.F.S.).

2.20 Convex Combination and Convex Sets

Convex combination

The convex combination of the finite set of points $X_1, X_2, \dots, X_m$ in E^n is a point X given by $X = \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_m X_m$, where λ_i's are non-negative real numbers for $i = 1, 2, \dots, m$ and $\sum_{i=1}^{m} \lambda_i = \lambda_1 + \lambda_2 + \dots \lambda_m = 1$.

The convex combination of two points is nothing but the line segment joining two points.

Convex Set

[CP 2011, '13, '15, '18]

A set of points X is said to be a convex set if the line segment joining any two distinct points of X is also in X. In otherwards if the convex combination of any two points of X is in X, then the set X is called a convex set.

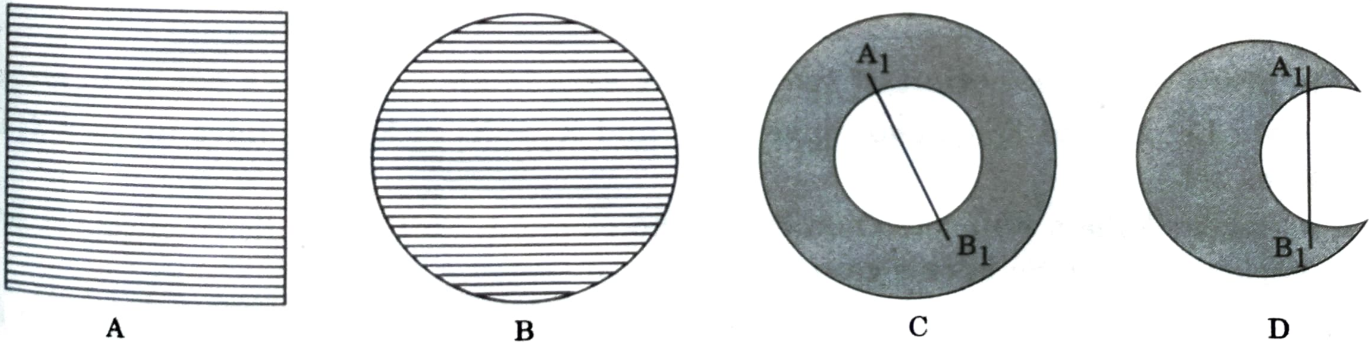


Fig. 2.1

In fig. 2.1, the sets A and B are convex set and C and D are not convex sets.

Note

13. **Show that $x_1 = 3, x_2 = 5, x_3 = 0$ is a basic solution of the system :**

$$2x_1 + x_2 + 4x_3 = 11$$
$$3x_1 + x_2 + 5x_3 = 14$$

[CP 2015, 2018]

Solution Clearly $x_1 = 3$, $x_2 = 5$, $x_3 = 0$ satisfy both the equations and hene (3, 5, 0) is a solution of this system.

The system of equations can be written in matrix form as

$$\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$$

The rank of the coefficient matrix is 2 $\left[\because \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} = 1 \neq 0\right]$

So the system has basic solution. Now $x_1 = 3$, $x_2 = 5$, $x_3 = 0$ will be a basic solution if the square sub-matrix corresponding to x_1 and x_2 is non-singular.

Now $\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 - 3 = -1 \neq 0$.

So the $x_1 = 3$, $x_2 = 5$, $x_3 = 0$ is a basic solution of the given system.

14. **Find a basic solution of the set of equations**

$$2x_1 + 3x_2 + x_3 = 8$$
$$x_1 + 2x_2 + 2x_3 = 5$$

[CP 2017]

Solution The given system of equations can be written in matrix form

as $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

Here the rank of the coefficient matrix is 2

$\left[\because \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0\right]$, which is equal to the number of equations.

So the above system of equations may have at most ${}^3C_2$ *i.e.*, 3 basic solutions.

$\because \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \neq 0$, so we can get a basic solution corresponding to $x_3 = 0$.

Putting $x_3 = 0$ in the system of equations, we get

$$2x_1 + 3x_2 = 8$$
$$x_1 + 2x_2 = 5$$

i.e., $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$ or, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \begin{pmatrix} 16-15 \\ -8+10 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore x_1 = 1$, $x_2 = 2$, $x_3 = 0$ is a basic solution of the given system of equations.

15. **Find all the basic solutions of the given set of equations.**

$$2x + 3y - 5z = 5$$

$$4x + 2y + 4z = 6$$

Mention which are basic feasible solutions among the basic solution.

[CP 2014, 2016]

Solution The given system of equations can be written in matrix form as

$$\begin{pmatrix} 2 & 3 & -5 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Here the rank of the coefficient matrix is 2

$[\because \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = 4 - 12 = -8 \neq 0]$, which is equal to the number of equations.

So the given system of equations may have ${}^3C_2$ *i.e.* 3 basic solutions.

Since $\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} \neq 0$, so we can get a basic solution corresponding to $z = 0$.

Putting $z = 0$ in the system, we get,

$$2x + 3y = 5$$

$$4x + 2y = 6$$

i.e., $\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$= \frac{1}{-8}\begin{pmatrix} 2 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{-8}\begin{pmatrix} 10-18 \\ -20+12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore x = 1, y = 1, z = 0$ is a basic solution and it is feasible.

Again, $\begin{vmatrix} 3 & -5 \\ 2 & 4 \end{vmatrix} = 12 + 10 = 22 \neq 0$, so we can get a basic solution corresponding to $x = 0$.

Putting $x = 0$ in the system, we get

$$3y - 5z = 5$$

$$2y + 4z = 6$$

i.e., $\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$$\therefore \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 20+30 \\ -10+18 \end{pmatrix}\begin{pmatrix} \frac{25}{11} \\ \frac{4}{11} \end{pmatrix}$$

$\therefore x = 0, y = \frac{25}{11}, z = \frac{4}{11}$ is a basic solution and it is feasible.

Again, $\begin{vmatrix} 2 & -5 \\ 4 & 4 \end{vmatrix} = 8 + 20 = 28 \neq 0$, so we can get a basic solution corresponding to $y = 0$.

Putting $y = 0$ in the system, we get,

$$2x - 5z = 5$$
$$4x + 4z = 6$$

i.e., $\begin{pmatrix} 2 & -5 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

or, $\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 4 & 5 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 20+30 \\ -20+12 \end{pmatrix}\begin{pmatrix} \frac{25}{14} \\ -\frac{2}{7} \end{pmatrix}$

$\therefore x = \frac{25}{14}, y = 0, z = -\frac{2}{7}$ is a basic solution and it is not feasible.

So the given system of equations has three basic solutions (1, 1, 0), $\left(0, \frac{25}{11}, \frac{4}{11}\right)$ and $\left(\frac{25}{14}, 0, -\frac{2}{7}\right)$ and first two are basic feasible solutions.

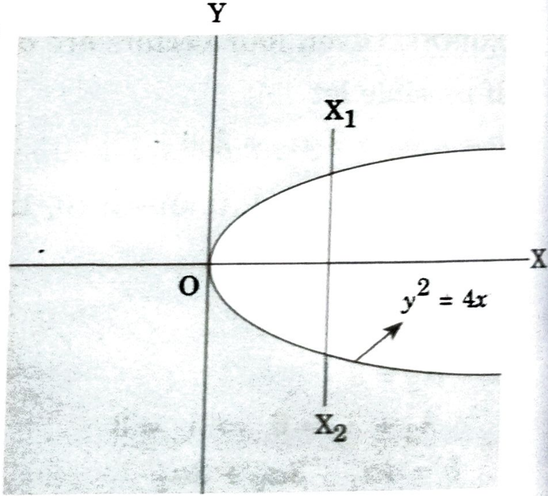
21. **Show that in E^2, the set $X = \{(x, y) : y^2 \geq 4x\}$ is not a convex set.**

[CP 2014, 2016]

Solution The set X is nothing but the set of all points in the *xy*-plane, which lies outside the parabola $y^2 = 4x$.

The shadded region, shown in the figure is the set X.

If we take two points X_1 and X_2 in the region as shown in the figure. The line segment joining X_1 and X_2 does not lie wholly within the region. So X is not a convex set.



23. **Prove that in E^2, the set $X = \{(x, y) \mid x + 2y \leq 5\}$ is a convex set.**

[CP 2008, 2012]

Solution Let $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ be any two points of X.

$\therefore x_1 + 2y_1 \leq 5$ and $x_2 + 2y_2 \leq 5$

Let $(x_3, y_3) = X_3 = \lambda X_1 + (1 - \lambda)X_2$ [where $0 \leq \lambda \leq 1$]

$= \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) = (\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)$

$\therefore x_3 = \lambda x_1 + (1 - \lambda)x_2$

$y_3 = \lambda y_1 + (1 - \lambda)y_2$

Now, $x_3 + 2y_3 = \{\lambda x_1 + (1 - \lambda)x_2\} + 2\{\lambda y_1 + (1 - \lambda)y_2\}$

$= \lambda(x_1 + 2y_1) + (1 - \lambda)(x_2 + 2y_2)$

$\leq \lambda \cdot 5 + (1 - \lambda)5$ [$\because x_1 + 2y_1 \leq 5$ and $x_2 + 2y_2 \leq 5$]

$= 5\lambda + 5 - 5\lambda = 5$

$\therefore x_3 + 2y_3 \leq 5$

Which implies $X_3 \in X$

i.e., the convex combination of any two points of X is also a point of X.

Hence X is a convex set.

1. **Draw the feasible region of the following inequations,**

$$2x + 3y \leq 6$$
$$x - y \leq 1$$
$$x, y \geq 0$$

[CP 2013, 2018]

Solution We first convert all the inequations to equations and draw their corresponding graphs.

$$2x + 3y = 6$$

$$\text{or, } \frac{x}{3} + \frac{y}{2} = 1 \qquad \text{... (1)}$$

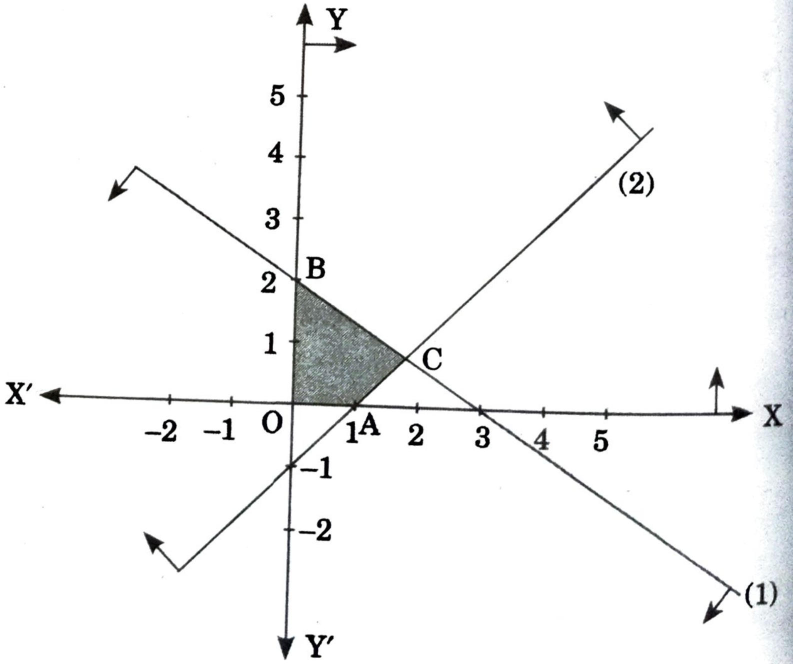
$$x - y = 1$$

$$\text{or, } \frac{x}{1} + \frac{y}{-1} = 1 \qquad \text{... (2)}$$

$$x = 0 \qquad \text{... (3)}$$

$$y = 0 \qquad \text{... (4)}$$

Now considering the inequations, we draw the feasible region (shaded region) OACBO, which is bounded.



5. **Solve graphically :**

Minimize $z = 300x_1 + 400x_2$

Subject to $6x_1 + 10x_2 \geq 60$

$$4x_1 + 4x_2 \geq 32$$

$$x_1, x_2 \geq 0$$

[CP 2014]

Solution We first convert all the inequations to equations and draw their corresponding graphs.

$$6x_1 + 10x_2 = 60 \text{ or, } \frac{x_1}{10} + \frac{x_2}{6} = 1 \quad \ldots (1)$$

$$4x_1 + 4x_2 = 32 \text{ or, } \frac{x_1}{8} + \frac{x_2}{8} = 1 \quad \ldots (2)$$

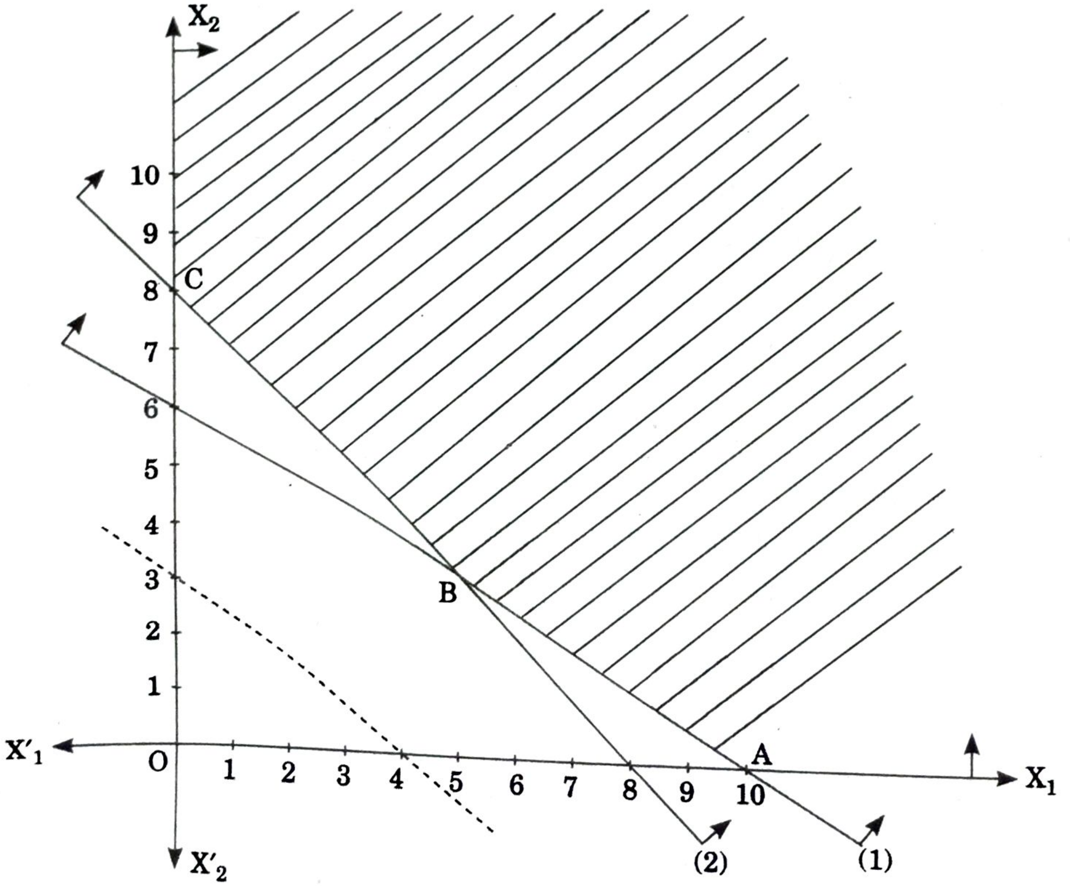
$$x_1 = 0 \quad \ldots (3)$$

$$x_2 = 0 \quad \ldots (4)$$

Now considering the inequations we draw the feasible space (shaded region), which is unbounded.

Let $300x_1 + 400x_2 = 1200$ (say)

$$\text{or, } \frac{x_1}{4} + \frac{x_2}{3} = 1 \quad \ldots (5)$$



We draw the graph of equation (5) by dotted line. Now we move the dotted line parallel to itself away from the origin. The first point of the feasible space will be the point of minimization. Here 'B' is the point of minimization. We get the coordinates of B by solving (1) and (2) which are (5, 3) and $z_{\min} = 300 \times 5 + 400 \times 3 = 1500 + 1200 = 2700$.

7. **Solve graphically :**

Maximize $z = 3x_1 + 2x_2$

Subject to $x_1 - x_2 \leq 2$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Solution We first convert all the inequations to equations and draw their corresponding graphs :

$$x_1 - x_2 = 2 \text{ or, } \frac{x_1}{2} + \frac{x_2}{-2} = 1 \quad \text{... (1)}$$

$$x_1 + x_2 = 6 \text{ or, } \frac{x_1}{6} + \frac{x_2}{6} = 1 \quad \text{(2)}$$

$$x_1 = 0 \quad \text{... (3)}$$

$$x_2 = 0 \quad \text{... (4)}$$

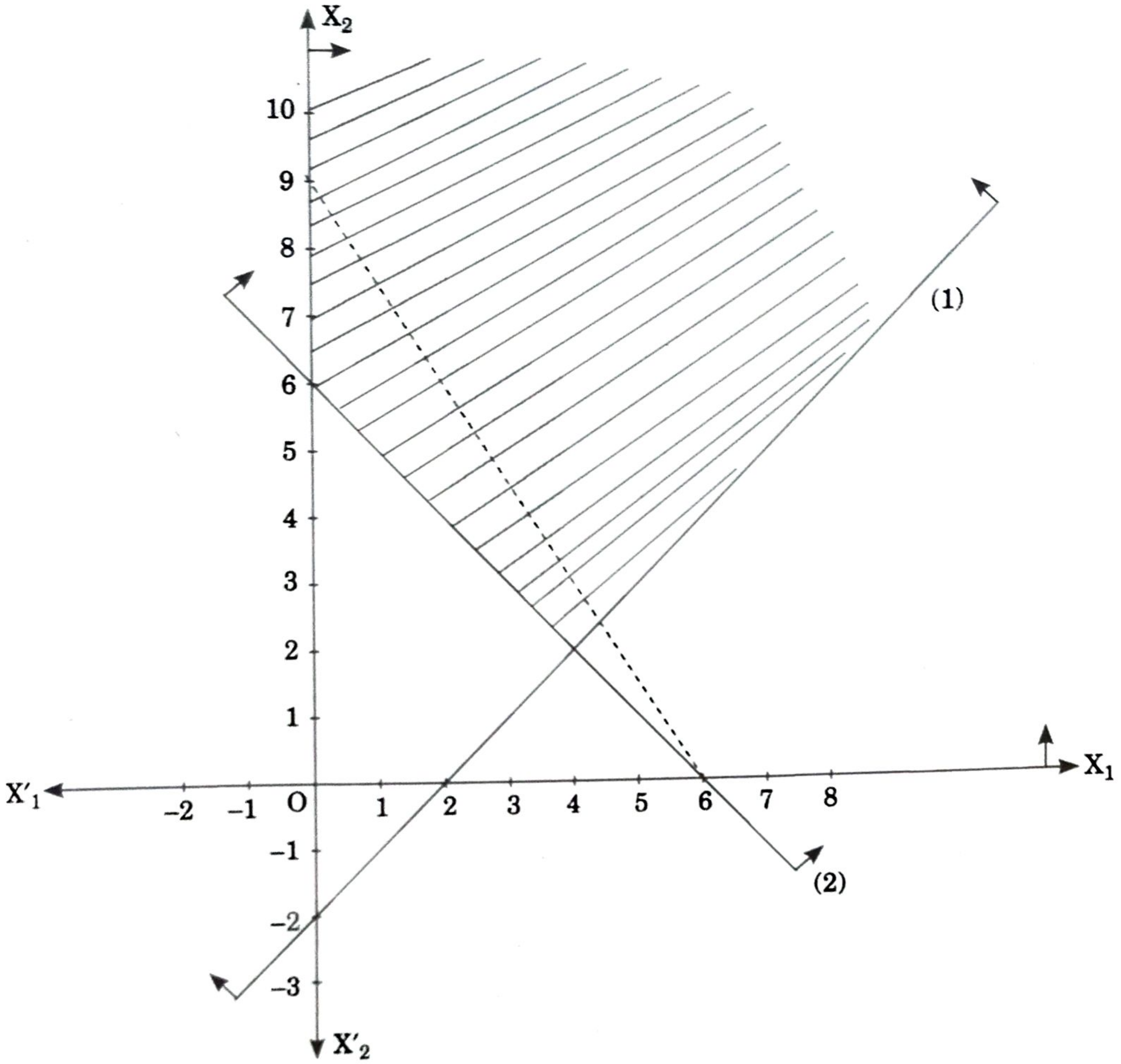
Now considering the inequations, we draw the feasible region (shaded region) which is unbounded.

Let $3x_1 + 2x_2 = 18$ (say)

$$\text{or, } \frac{x_1}{6} + \frac{x_2}{9} = 1 \quad \text{... (5)}$$

We draw the graph of equation (5) by dotted line. Since the problem is of maximization type, we have to move the line indefinitely away from the origin but no finite maximum value of z can be achieved within the feasible region.

Thus the given linear programming problem has unbounded solution.



Note

In this case the feasible region is unbounded and the problem has unbounded solution.

8. **Solve graphically :**
Maximize $z = 4x_1 + 2x_2$
Subject to $2x_1 + x_2 \leq 1$
$3x_1 + 4x_2 \geq 6$
$x_1, x_2 \geq 0$

Solution We first convert all the inequations to equations and draw their corresponding graphs.

$$2x_1 + x_2 = 1 \text{ or, } \frac{x_1}{\frac{1}{2}} + \frac{x_2}{1} = 1 \qquad \text{... (1)}$$

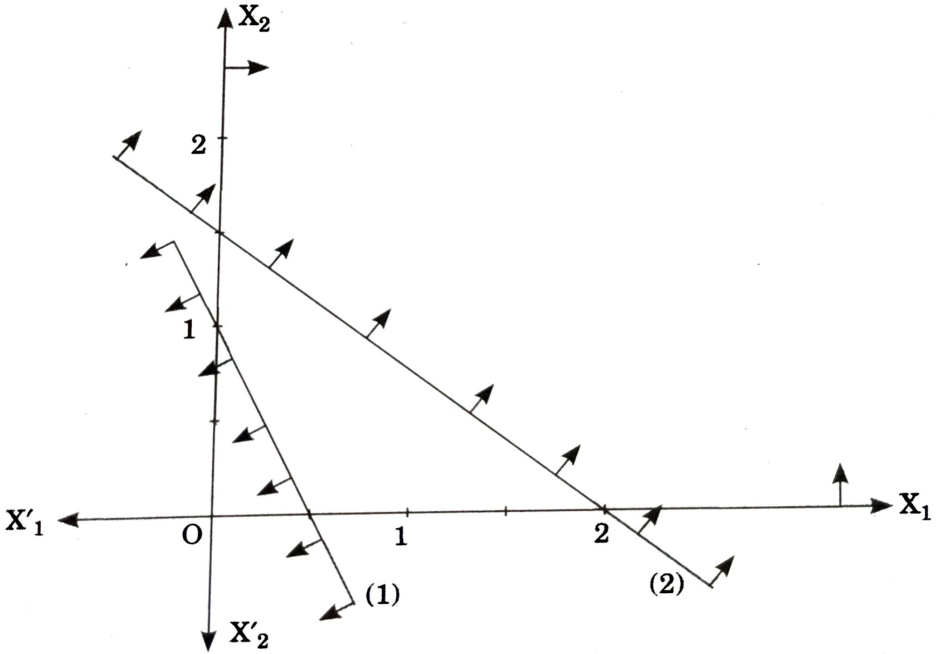
$$3x_1 + 4x_2 = 6 \text{ or } \frac{x_1}{2} + \frac{x_2}{\frac{3}{2}} = 1 \qquad \text{... (2)}$$

$$x_1 = 0 \qquad \text{... (3)}$$

$$x_2 = 0 \qquad \text{... (4)}$$

Now considering the inequations, we have tried to draw the feasible space but failed. And consequently no feasible solution is possible.

Hence the given linear programming problem has no feasible solution.



Note

In this case there exist no feasible region and so the problem has no feasible solution.

3. **Show that (0, 2, 3, 4) is a basic feasible solution of the set of equations**

$$7x_1 + x_2 + 3x_3 + 3x_4 = 23$$
$$7x_1 + 2x_2 + 4x_3 + x_4 = 20$$
$$14x_1 + 3x_2 + 2x_3 + 9x_4 = 48$$

whose feasible solution is (1, 1, 2, 3).

Solution Here the number of variables is four but the number of equations is three. So a basic solution of this system has at most three non-zero variables.

Now we rewrite the system of equations as

$$\boldsymbol{a}_1x_1 + \boldsymbol{a}_2x_2 + \boldsymbol{a}_3x_3 + \boldsymbol{a}_4x_4 = \boldsymbol{b} \quad \text{... (1)}$$

where $\boldsymbol{a}_1 = \begin{pmatrix} 7 \\ 7 \\ 14 \end{pmatrix}$, $\boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\boldsymbol{a}_3 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$, $\boldsymbol{a}_4 = \begin{pmatrix} 3 \\ 1 \\ 9 \end{pmatrix}$, and $\boldsymbol{b} = \begin{pmatrix} 23 \\ 20 \\ 48 \end{pmatrix}$

Since (1, 1, 2, 3) is a feasible solution of the given system of equations, so from (1), we get

$$\boldsymbol{a}_1 + \boldsymbol{a}_2 + 2\boldsymbol{a}_3 + 3\boldsymbol{a}_4 = \boldsymbol{b} \quad \text{... (2)}$$

Now the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ are clearly linearly dependent set of vectors (since at most three vectors can be linearly independent in E^3).

Let $\lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \lambda_3\boldsymbol{a}_3 + \lambda_4\boldsymbol{a}_4 = \boldsymbol{0}$ ... (3)

where at least one λ_i $(i = 1, 2, 3, 4)$ is non-zero. Equation (3) gives

$$\lambda_1\begin{pmatrix} 7 \\ 7 \\ 14 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_3\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda_4\begin{pmatrix} 3 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives,

$$7\lambda_1 + \lambda_2 + 3\lambda_3 + 3\lambda_4 = 0 \quad \text{... (4)}$$
$$7\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 = 0 \quad \text{... (5)}$$
$$14\lambda_1 + 3\lambda_2 + 2\lambda_3 + 9\lambda_4 = 0 \quad \text{... (6)}$$

$(5) - (4) \Rightarrow \lambda_2 + \lambda_3 - 2\lambda_4 = 0$ ... (7)

$(6) - 2 \times (4) \Rightarrow \lambda_2 - 4\lambda_3 + 3\lambda_4 = 0$ ... (8)

Solving (7) and (8), we get

$$\frac{\lambda_2}{3-8} = \frac{\lambda_3}{-2-3} = \frac{\lambda_4}{-4-1}$$

or, $\dfrac{\lambda_2}{1} = \dfrac{\lambda_3}{1} = \dfrac{\lambda_4}{1} = k$ (say)

$\therefore \lambda_2 = \lambda_3 = \lambda_4 = k$

From (4) we get, $7\lambda_1 = -\lambda_2 - 3\lambda_3 - 3\lambda_4 = -k - 3k - 3k = -7k$

$\Rightarrow \lambda_1 = -k$

We take $k = -1$ [$\because$ we have to show (0, 2, 3, 4) is a basic feasible solution].

So $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = -1$

Then (3) becomes

$$\boldsymbol{a}_1 - \boldsymbol{a}_2 - \boldsymbol{a}_3 - \boldsymbol{a}_4 = \boldsymbol{0} \quad \text{... (9)}$$

To reduce the number of positive variables of the given solution, we use the following criterion to determine the vectors to be eliminated for $\lambda_j > 0$.

$$\frac{x'_r}{\lambda_r} = \operatorname*{Min}_{j}\left\{\frac{x'_j}{\lambda_j},\ \lambda_j > 0\right\} = \text{Min}\ \left\{\frac{x'_1}{\lambda_1}\right\} = \frac{1}{1} = 1$$

where x'_i is the value of x_i in the given feasible solution.

Now we have to eliminate $\boldsymbol{a}_1$. From (9) $\boldsymbol{a}_1 = \boldsymbol{a}_2 + \boldsymbol{a}_3 + \boldsymbol{a}_4$

From (2) $0 \cdot \boldsymbol{a}_1 + 2\boldsymbol{a}_2 + 3\boldsymbol{a}_3 + 4\boldsymbol{a}_4 = \boldsymbol{b}$ ... (10)

Comparing (1) and (10), we get (0, 2, 3, 4) is a basic feasible solution of the system.

5. **Solve the following linear programming by simplex method.**

Maximize $z = -x_1 + 3x_2 - 2x_3$

Subject to

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 &\le 7\\
-2x_1 + 4x_2 \quad &\le 12\\
-4x_1 + 3x_2 + 8x_3 &\le 10\\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

[CP 2009, 2011, 2014, 2015]

Solution Introducing slack variables x_4, x_5, x_6 we rewrite the problem as

Max $z = -x_1 + 3x_2 - 2x_3 + 0.x_4 + 0\cdot x_5 + 0\cdot x_6$

Subject to

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 + x_4 \qquad &= 7\\
-2x_1 + 4x_2 \qquad + x_5 \quad &= 12\\
-4x_1 + 3x_2 + 8x_3 \qquad + x_6 &= 10\\
x_i \ge 0,\ i = 1, 2, 3, 4, 5, 6
\end{aligned}
$$

The initial basic feasible solution is [0, 0, 0, 7, 12, 10] and pass on to the first simplex tableau as follows :

Tableau-1

			c_j	-1	3	-2	0	0	0	
C_B	B	X	b	a_1	a_2	a_3	a_4	a_5	a_6	Mini ratio
0	a_4	x_4	7	3	-1	2	1	0	0	—
0	a_5	x_5	12	-2	④	0	0	1	0	$\frac{12}{4} = 3 \rightarrow$
0	a_6	x_6	10	-4	3	8	0	0	1	$\frac{10}{3}$
$z = 0$		$z_j - c_j$		1	-3	2	0	0	0	
					↑			↓		

Here $\boldsymbol{a}_2$ is the entering and $\boldsymbol{a}_5$ is the departing vector. Now we pass to second tableau.

Tableau-2

			c_j	-1	3	-2	0	0	0	Mini ratio
C_B	**B**	**X**	b	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	
0	$\boldsymbol{a}_4$	x_4	10	$\boxed{\frac{5}{2}}$	0	2	1	$\frac{1}{4}$	0	$\frac{10}{\frac{5}{2}} = 4$
3	$\boldsymbol{a}_2$	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	—
0	$\boldsymbol{a}_6$	x_6	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	
$z = 9$		$z_j - c_j$		$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	
				↑			↓			

Here $\boldsymbol{a}_1$ is the entering vector and $\boldsymbol{a}_4$ is the departing vector. So we pass to third tableau.

Tableau-3

			c_j	-1	3	-2	0	0	0
C_B	**B**	**X**	b	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$
-1	$\boldsymbol{a}_1$	x_1	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0
3	$\boldsymbol{a}_2$	x_2	5	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0
0	$\boldsymbol{a}_6$	x_6	11	0	0	10	1	$-\frac{1}{2}$	1
$z = 11$		$z_j - c_j$		0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0

Here all $z_j - c_j \geq 0$, so the optimal stage is reached and $z_{\max} = (-1) \times 4 + 3 \times 5 = 11$ at $x_1 = 4$, $x_2 = 5$, $x_3 = 0$

[In the solution, only given variables are to be mentioned]

9. **Solve the following linear programing by simplex method**

Maximize $z = 2x_2 + x_3$

Subject to $x_1 + x_2 - 2x_3 \le 7$

$-3x_1 + x_2 + 2x_3 \le 3$

$x_1, x_2, x_3 \ge 0$

Solution Introducing slack variables x_4 and x_5, we rewrite the problem as

Maximize $z = 0.x_1 + 2x_2 + x_3 + 0.x_4 + 0.x_5$

Subject to $x_1 + x_2 - 2x_3 + x_4 = 7$

$-3x_1 + x_2 + 2x_3 + x_5 = 3$

$x_i \ge 0, i = 1, 2, 3, 4, 5$

The initial basic feasible solution is [0, 0, 0, 7, 3] and we pass to tableau-1.

Tableau-1

			c_j	0	2	1	0	0	Mini ratio
C_B	**B**	**X**	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	
0	$\boldsymbol{a_4}$	x_4	7	1	1	–2	1	0	7
0	$\boldsymbol{a_5}$	x_5	3	–3	①	2	0	1	3 →
$z = 0$		$z_j - c_j$		0	–2	–1	0	0	
					↑			↓	

Here $\boldsymbol{a_2}$ is the entering vector and $\boldsymbol{a_5}$ is the departing vector and we pass to tableau-2

Tableau-2

			c_j	0	2	1	0	0
C_B	**B**	**X**	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
0	$\boldsymbol{a_4}$	x_4	4	–4	0	–4	1	–1
2	$\boldsymbol{a_2}$	x_2	3	–3	1	2	0	1
$z = 6$		$z_j - c_j$		–6	0	4	0	2

Here only $z_1 - c_1 = -6 < 0$, So $\boldsymbol{a_1}$ is the entering vector. But all $y_{i1} < 0$ *i.e.*, y_{11}, y_{21} are negative. Hence the problem has no finite optimum value of the objective function. Hence the problem has unbounded solution.

10. **Solve the following linear programming problem**

Maximize $z = x_1 + 3x_2 + 2x_3$

Subject to $x_1 + 2x_2 \le 10$

$2x_1 + x_3 \le 8$

$2x_2 + x_3 \le 6$

and $x_1, x_2, x_3 \ge 0$

[CP 2017]

Solution Introducing slack variables x_4, x_5, x_6, we rewrite the problem as

$$\begin{aligned}
&\text{Maximize } z = x_1 + 3x_2 + 2x_3 + 0{\cdot}x_4 + 0{\cdot}x_5 + 0{\cdot}x_6\\
&\text{Subject to } x_1 + 2x_2 + x_4 = 10\\
&\qquad 2x_1 + x_3 + x_5 = 8\\
&\qquad 2x_2 + x_3 + x_6 = 6\\
&\qquad x_i \geq 9,\ i = 1, 2, \ldots, 6
\end{aligned}$$

The initial basic feasible solution is [0, 0, 0, 10, 8, 6] and we pass to tableau-1.

Tableau-1

			c_j	1	3	2	0	0	0	Mini ratio
C_B	B	X	b	a_1	a_2	a_3	a_4	a_5	a_6	
0	a_4	x_4	10	1	2	0	1	0	0	5
0	a_5	x_5	8	2	0	1	0	1	0	—
0	a_6	x_6	6	0	②	1	0	0	1	3 →
$z = 0$		$z_j - c_j$		−1	−3	−2	0	0	0	
					↑				↓	

a_2 is the entering vector and a_6 is the departing vector and we pass to tableau-2

Tableau-2

			c_j	1	3	2	0	0	0	Mini ratio
C_B	B	X	b	a_1	a_2	a_3	a_4	a_5	a_6	
0	a_4	x_4	4	①	0	−1	1	0	−1	4 →
0	a_5	x_5	8	2	0	1	0	1	0	4
3	a_2	x_2	3	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	—
$z = 9$		$z_j - c_j$		−1	0	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	
				↑			↓			

a_1 is the entering vector and a_4 is the departing vector and we pass to tableau-3.

Tableau-3

			c_j	1	3	2	0	0	0	Mini ratio
C_B	B	X	b	a_1	a_2	a_3	a_4	a_5	a_6	
1	a_1	x_1	4	1	0	−1	1	0	−1	—
0	a_5	x_5	0	0	0	③	−2	1	2	0 →
3	a_2	x_2	3	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	6
$z = 13$		$z_j - c_j$		0	0	$-\frac{3}{2}$	1	0		
						↑		↓		

Here $\boldsymbol{a}_3$ is the entering vector and $\boldsymbol{a}_5$ is the departing vector and we pass to tableau-4.

Tableau-4

			c_j	1	3	2	0	0	0	Mini ratio
C_B	**B**	**X**	$\boldsymbol{b}$	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	
1	$\boldsymbol{a}_1$	x_1	4	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	12
2	$\boldsymbol{a}_3$	x_3	0	0	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	—
3	$\boldsymbol{a}_2$	x_2	3	0	1	0	$\enclose{circle}{\frac{1}{3}}$	$-\frac{1}{6}$	$\frac{1}{6}$	9
$z = 13$		$z_j - c_j$		0	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$	
					↓		↑			

Here all $z_j - c_j \geq 0$, so the optimal stage is reached and the optimal solution is $x_1 = 4$, $x_2 = 3$, $x_3 = 0$ and $z_{\max} = 1 \times 4 + 3 \times 3 + 2 \times 0 = 13$

But $z_4 - c_4 = 0$ for non-basic variable x_4 and y_{14} and y_{34} are positive. This implies that the given problem has infinite number of solutions.

15. **Use Penalty method to solve the following linear programming problem :**

Maximize $z = 5x_1 + 11x_2$

Subject to $2x_1 + x_2 \leq 4$

$$3x_1 + 4x_2 \geq 24$$

$$2x_1 - 3x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Solution Introducing slack variable x_3, surplus variables x_4, x_5 and artificial variables x_6, x_7, we rewrite the problem as.

Maximize $z = 5x_1 + 11x_2 + 0{\cdot}x_3 + 0{\cdot}x_4 + 0{\cdot}x_5 - Mx_6 - Mx_7$

Subject to $2x_1 + x_2 + x_3 = 4$

$$3x_1 + 4x_2 - x_4 + x_6 = 24$$

$$2x_1 - 3x_2 - x_5 + x_7 = 6$$

$$x_i \geq 0,\ i = 1, 2, \ldots, 7$$

Where M is a very large positive number.

Here [0, 0, 4, 0, 0, 24, 6] is the initial basic feasible solution. We pass to tableau-1.

Tableau-1

			c_j	5	11	0	0	0	–M	–M	Mini ratio
C_B	B	X	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	
0	a_3	x_3	4	②	1	1	0	0	0	0	2 →
–M	a_6	x_6	24	3	4	0	–1	0	1	0	8
–M	a_7	x_7	6	2	–3	0	0	–1	0	1	3
$z = -30M$			$z_j - c_j$	–5M–5	–M–11	0	M	M	0	0	
				↑		↓					

Tableau-2

			c_j	5	11	0	0	0	–M	–M
$\mathbf{C_B}$	**B**	**X**	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$
5	$\boldsymbol{a_1}$	x_1	2	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
–M	$\boldsymbol{a_6}$	x_6	18	0	$\frac{5}{2}$	$-\frac{3}{2}$	–1	0	1	0
–M	$\boldsymbol{a_7}$	x_7	2	0	–4	–1	0	–1	0	1
$z = -20M = 10$			$z_j - c_j$	0	$\frac{3}{2}M - \frac{17}{2}$	$\frac{5}{2}M + \frac{5}{2}$	M	M	0	0

Here all $(z_j - c_j) \geq 0$, but there are two vectors corresponding to the artificial variables lie in the basis and values of the artificial variables are positive, so the given linear programing problem does not possess any feasible solution *i.e.*, the problem has no feasible solution.

Primal Problem :

Maximize $z = \mathbf{CX}$

Subject to $\mathbf{AX} \le \boldsymbol{b}$

$\mathbf{X} \ge \mathbf{0}$

Dual Problem :

Minimize $w = \boldsymbol{b}^{\mathrm{T}}\mathbf{V}$

Subject to $\mathbf{A}^{\mathrm{T}}\mathbf{V} \ge \mathbf{C}^{\mathrm{T}}$

$\mathbf{V} \ge \mathbf{0}$

Where $\mathrm{A} = (a_{ij})_{m \times n}$, $i = 1, 2, ..., m; j = 1, 2, ... , n$

$\mathbf{C} = (c_1, c_2, ..., c_n)$, $\boldsymbol{b} = [b_1, b_2, ..., b_m]$, $\mathbf{X} = [x_1, x_2, ..., x_n]$

$\mathbf{V} = [v_1, v_2, ..., v_m]$ and A^{T}, $\boldsymbol{b}^{\mathrm{T}}$ and $\mathbf{C}^{\mathrm{T}}$ are the corresponding transpose matrices.

The four possible cases in primal dual problem as follows :

Primal	Dual	Conclusion
Feasible solution	Feasible solution	Finite optimal values of both exist.
Feasible solution	No Feasible solution	The Primal has unbounded solution
No Feasible solution	Feasible solution	Unbounded solution of the dual
No Feasible solution	No Feasible solution	No optimal solution of the either problem

8. **Use duality to find the optimal solution (if any) of the following Linear Programming Problem :**

Minimize $z = 15x_1 + 10x_2$

Subject to $3x_1 + 5x_2 \geq 5$

$5x_1 + 2x_2 \geq 3$

$x_1, x_2 \geq 0$

Solution Given problem is a standard minimization problem. We take v_1, v_2 as dual variables. The dual of the given problem is

Max $w = 5v_1 + 3v_2$

Subject to $3v_1 + 5v_2 \leq 15$

$5v_1 + 2v_2 \leq 10$

$v_1, v_2 \geq 0$

Now we will solve the problem by simplex method.

Introducing slack variables v_3, v_4, we rewrite the problem as

Max $w = 5v_1 + 3v_2 + 0 \cdot v_3 + 0 \cdot v_4$

Subject to $3v_1 + 5v_2 + v_3 = 15$

$5v_1 + 2v_2 + v_4 = 10$

$v_1, v_2, v_3, v_4 \geq 0$

The initial basic feasible solution is [0, 0, 15, 10] and we pass to tableau-1.

Tableau-1

			c_j	5	3	0	0	
C_B	B	X	b	a_1	a_2	a_3	a_4	Mini ratio
0	a_3	v_3	15	3	5	1	0	5
0	a_4	v_4	10	⑤	2	0	1	2 →
	$z = 0$		$z_j - c_j$	− 5	− 3	0	0	
				↑			↓	

Tableau-2

			c_j	5	3	0	0	
C_B	**B**	**X**	***b***	a_1	a_2	a_3	a_4	
0	a_3	v_3	9	0	$\boxed{\frac{19}{5}}$	1	$-\frac{3}{5}$	$\frac{45}{19}$ →
5	a_1	v_1	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	5
$z = 10$			$z_j - c_j$	0	-1	0	1	
					↑	↓		

Tableau-3

			c_1	5	3	0	0
C_B	**B**	**X**	***b***	a_1	a_2	a_3	a_4
3	a_2	v_2	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$-\frac{3}{19}$
5	a_1	v_1	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{19}$
$z = \frac{235}{19}$			$z_j - c_j$	0	0	$\frac{5}{19}$	$\frac{16}{19}$

Since $z_j - c_j \geq 0$ for all j; so the optimal stage is reached and $w_{\max} = \frac{235}{19}$ at $v_1 = \frac{20}{19}$ and $v_2 = \frac{45}{19}$

By the duality theory, we get the optimal value of the objective function of the primal as $\frac{235}{19}$ *i.e.*, $z_{\min} = \frac{235}{19}$ at $x_1 = \frac{5}{19}$, $x_2 = \frac{16}{19}$.

[We get the values of x_1, x_2 from the net evaluation corresponding to the slack variables v_3, v_4 in the final table]

10. **Obtain the dual of the following linear programming problem and hence solve it.**

Maximize $z = 3x_1 + 4x_2$

Subject to $x_1 - x_2 \le 1$

$x_1 + x_2 \ge 4$

$x_1 - 3x_2 \le 3$

$x_1, x_2 \ge 0$

Solution We first rewrite the given problem in standard form as

Maximize $z = 3x_1 + 4x_2$

Subject to $x_1 - x_2 \le 1$

$-x_1 - x_2 \le -4$

$x_1 - 3x_2 \le 3$

$x_1, x_2 \ge 0$

The dual of the above problem is

Min $w = v_1 - 4v_2 + 3v_3$

Subject to $v_1 - v_2 + v_3 \ge 3$

$-v_1 - v_2 - 3v_3 \ge 4$

$v_1, v_2, v_3 \ge 0$

Where v_1, v_2, v_3 are dual variables.

Now we maximize w^* $(= -w)$ under same constraints.

Introducing surplus variables v_4, v_5 and artificials variable v_6, v_7, we rewrite the problem as,

Max $w^* = -v_1 + 4v_2 - 3v_3 + 0.v_4 + 0 \cdot v_5 - Mv_6 - Mv_7$

Subject to $v_1 - v_2 + v_3 - v_4 + v_6 = 3$

$-v_1 - v_2 - 3v_3 - v_5 + v_7 = 4$

$v_i \ge 0, i = 1, 2, \ldots, 7$

Where M is a very large real number.

Here [0, 0, 0, 0, 0, 3, 4] is the initial basic feasible solution and we pass to tableau-1.

Tableau-1

			c_j	-1	4	-3	0	0	$-M$	$-M$
C_B	**B**	**X**	$\boldsymbol{b}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$
$-M$	$\boldsymbol{a_6}$	v_6	3	1	-1	1	-1	0	1	0
$-M$	$\boldsymbol{a_7}$	v_7	4	-1	-1	-3	0	-1	0	1
$w^* = -7M$			$z_j - c_j$	1	$2M - 4$	$2M + 3$	M	M	0	0

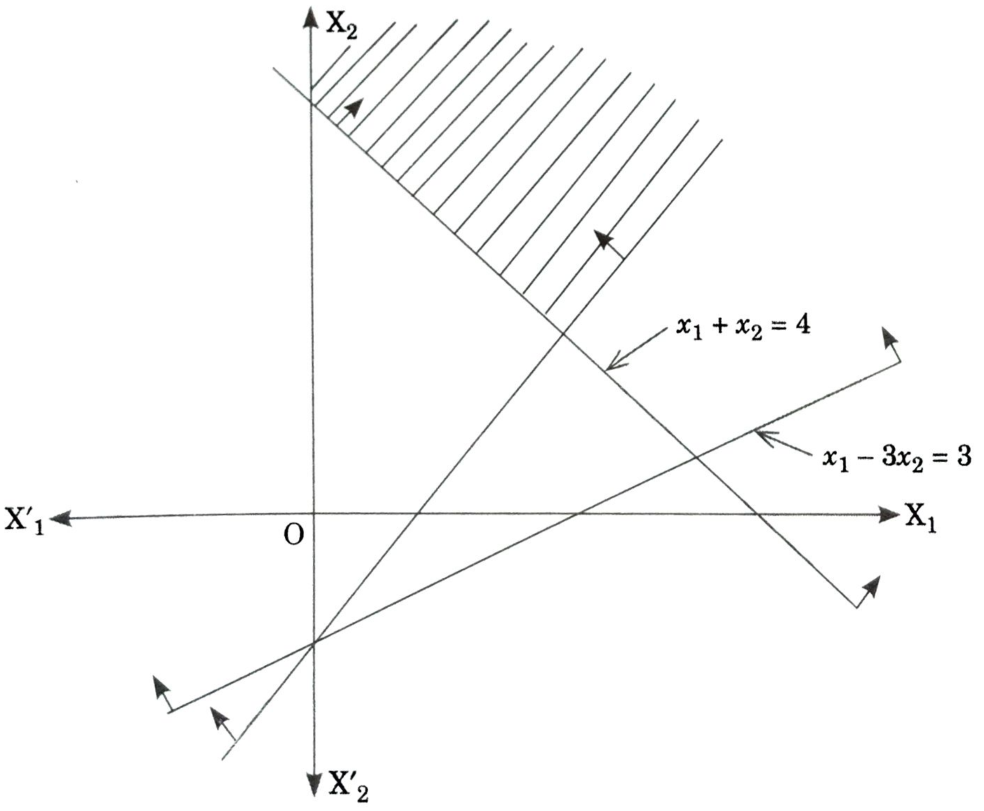
Here in the initial table all $z_j - c_j \geq 0$, but the artificial variables are in the basis, so the dual problem has no feasible solution.

Now there are two possibilities for the solutions of the primal problem :

(i) The primal has no feasible solution.

(ii) If the primal has feasible solution, then it has unbounded solution.

Now we use graphical method to find the feasible space (if any) for the primal problem.



Here we get feasible space, so the primal problem has unbounded solution.

1. **Find an initial basic feasible solution of the following transportation problem by North-West Corner method.**

	D_1	D_2	D_3	D_4	
O_1	9	8	5	7	12
O_2	4	6	8	7	14
O_3	5	8	9	5	16
	8	18	13	3	

[CP 2015, 2019]

Solution

	D_1	D_2	D_3	D_4	
O_1	$\boxed{8}$ 9	$\boxed{4}$ 8	5	7	~~12~~ ~~4~~
O_2	4	$\boxed{14}$ 6	$\boxed{0}$ 8	7	~~14~~
O_3	5	8	$\boxed{13}$ 9	$\boxed{3}$ 5	~~16~~ ~~3~~
	~~8~~	~~18~~ ~~14~~	~~13~~	~~3~~	

Here total supply = 12 + 14 + 16 = 42

Total demand = 8 + 18 + 13 + 3 = 42

So the problem is balanced.

Now we try to find an initial basic feasible solution of the given transportation problem.

We get the solution

$x_{11} = 8, x_{12} = 4, x_{22} = 14, x_{23} = 0, x_{33} = 13, x_{34} = 3$

and the corresponding cost = 72 + 32 + 84 + 0 + 117 + 15

= 320 units.

The initial basic feasible solution is degenerate. [Since number of positive allotments is 5 < 3 + 4 − 1]

Vogel's Approximation Method (VAM)

	D_1	D_2	D_3	D_4				
O_1	$\boxed{20}$ 1	2	$\boxed{10}$ 1	4	30(0)	30(0)	30(0)	10(1)
O_2	3	$\boxed{20}$ 3	$\boxed{20}$ 2	$\boxed{10}$ 1	50(1)	40(1)	40(1)	40(1)
O_3	4	$\boxed{20}$ 2	5	9	20(2)	20(2)		
	20(2)	40(0)	30(1)	10(3)				
	20(2)	40(0)	30(1)					
	20(2)	20(1)	30(1)					
		20(1)	30(1)					

Ex. 4. *Obtain an optimal basic feasible solution to the following transportation problem :*

	W_1	W_2	W_3	W_4	
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
	5	8	7	14	

[*Kalyani M. Sc., 1982; Vidyasagar Hons., 2002*]

We use *VAM* to find as usual the initial basic feasible solution. As seen from the table we see that the initial allocations are

$x_{11} = 5,\ x_{14} = 2,\ x_{23} = 7,\ x_{24} = 2,\ x_{32} = 8,\ x_{34} = 10,$

whose number is $m + n - 1 = 6$ and whose cost comes out as 779.

To test the optimality, we compute u_i and v_j and have the cell evaluations in circles of the unoccupied cells.

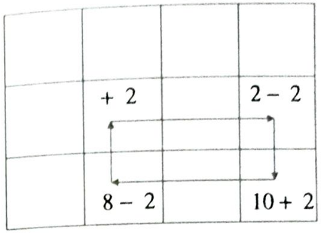
[The numbers in circles are the cell evaluations, that is, $c_{ij} - (u_i + v_j)$.]

	W_1	W_2	W_3	W_4	u_i
F_1	5 19	㉜ 30	(60) 50	2 10	0
F_2	① 70	(−18) 30	7 40	2 60	50
F_3	⑪ 40	8 8	(70) 70	10 20	10
v_j	19	− 2	− 10	10	

This shows that $\Delta_{22} = - 18$, a negative quantity. Hence this allocation does not give an optimal solution.

We allocate maximum quantity (+ 2) to this cell (2, 2) and make the cell (2, 4) empty. Re-adjustment of allocations are made to maintain the row capacities and column requirements. For that, we add 2 to the cell (2, 2), subtract 2 from (2, 4), add 2 to (3, 4) and subtract 2 from (3, 2), (shown in the tables in the next page).

With these revised allocations, the new transportation table is constructed and shown on the right below.



5 19	30	50	2 10	7
70	2 30	7 40	60	9
40	6 8	70	12 20	18
5	8	7	14	

In the next iteration the same procedure is followed. We compute u_i and v_j for the table and get all cell evaluations non-negative in the unoccupied cells. The allocated cells or any sub-set of them do not form a loop and hence the solution is basic.

The elements in circles are cell evaluations, that is, $\{c_{ij} - (u_i + v_j)\}$ and they are all positive.

	W_1	W_2	W_3	W_4	u_i
F_1	5 19	㉜ 30	㊷ 50	2 10	0
F_2	⑲ 70	2 30	7 40	⑱ 60	32
F_3	⑪ 40	6 8	(52) 70	12 20	10
v_j	19	− 2	8	10	

Thus the unique optimal solution is

$x_{11} = 5,\ x_{14} = 2,\ x_{22} = 2,\ x_{23} = 7,\ x_{32} = 6,\ x_{34} = 12$

and the optimal cost is

$5 \times 19 + 2 \times 10 + 2 \times 30 + 7 \times 40 + 6 \times 8 + 12 \times 20 = 743.$

Ex. 6. *Solve the following balanced transportation problem :*

	D_1	D_2	D_3	a_i
O_1	8	7	3	60
O_2	3	8	9	70
O_3	11	3	5	80
b_j	50	80	80	

[*Kalyani Hons., 1987*]

To find the initial basic solution, we use *VAM* and see that the initial solution is

$$x_{13} = 60,\ x_{21} = 50,\ x_{23} = 20,\ x_{32} = 80.$$

	D_1	D_2	D_3	a_i
O_1	8	7	60 3	60
O_2	50 3	8	20 9	70
O_3	11	80 3	5	80
b_j	50	80	80	

This solution shows that the number of occupied cells, that is, allocations is four which is not equal to $m + n - 1 = 5$. Hence the solution is degenerate.

To resolve this degeneracy, we add a small positive quantity ε to a cell such that this does not result in forming a loop among some or all of the occupied cells and make them dependent. For a dependent set of cells, unique determination of u_i and v_j will not be possible. With this in view, we allocate ε to the cell (1, 2) and construct the next

tableaux. According to the general rule, we compute u_i and v_j. The figures in circles in the unoccupied cells give $(u_i + v_j)$ for that cell.

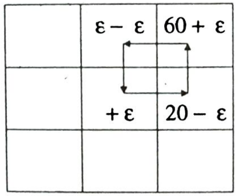
	D_1	D_2	D_3		u_i
O_1	(−3) 8	ε 7	60 3	60	0
O_2	50 3	(13) 8	20 9	70	6
O_3	(−7) 11	80 3	(−1) 5	80	− 4
	50	80	80		
v_j	− 3	7	3		

CELL EVALUATIONS

11	•	•
•	− 5	•
18	•	6

$\Delta_{22} = -5 < 0$

The cell evaluations $\{c_{ij} - (u_i + v_j)\}$ are shown in the above table on the right . Since, for the cell (2, 2), cell evaluation is negative, we allocate maximum possible unit to (2, 2) cell and adjust this additional allotment such that the cell (1, 2) becomes empty.



	D_1	D_2	D_3		u_i
O_1	(−3) 8	(2) 7	60 3	60	− 6
O_2	50 3	ε 8	20 9	70	0
O_3	(−2) 11	80 3	(4) 5	80	− 5
	50	80	80		
v_j	3	8	9		

We compute u_i and v_j again for the new allocation and show in circles $(u_i + v_j)$ of the unoccupied cells.

In the following table we show the cell evaluations $\{c_{ij} - (u_i + v_j)\}$ for the unoccupied cells.

CELL EVALUATIONS

11	5	•
•	•	•
13	•	1

All the cell evaluations being non-negative, we have the optimal solution. The occupied cells are basic cells.

Hence making $\varepsilon \to 0$, we get the optimal solution as

$$x_{13} = 60,\ x_{21} = 50,\ x_{23} = 20,\ x_{32} = 80.$$

The minimum cost is

$$60 \times 3 + 50 \times 3 + 20 \times 9 + 80 \times 3 = 750.$$

4. **Find the optimal assignment for the assignment problem with the following cost matrix :**

	W_1	W_2	W_3	W_4
M_1	1	4	6	3
M_2	9	7	10	9
M_3	4	5	11	7
M_4	8	7	8	5

[CP 2014, 2017]

Solution Given assignment cost matrix is a square matrix. So the problem is balanced.

We first subtract least element of each row from all the elements of that row and we get the reduced matrix as given below :

	W_1	W_2	W_3	W_4
M_1	0	3	5	2
M_2	2	0	3	2
M_3	0	1	7	3
M_4	3	2	3	0

So we connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is four, which is less than the order of the matrix. So the optimal stage is not reached. Now we select '3' which is the least of all surviving (uncrossed) elements. We subtract '3' from all the surviving elements and add it to the elements lying in the junction (The point of intersection of the two crossed out lines). Then the reduced matrix is given below :

	J_1	J_2	J_3	J_4	J_5
M_1	5	$\boxed{0}$	5	10	8
M_2	0	6	12	0	$\boxed{0}$
M_3	11	8	$\boxed{0}$	3	0
M_4	$\boxed{0}$	6	1	2	4
M_5	3	5	3	$\boxed{0}$	5

We again connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is 5, which is equal to the order of the matrix. So the optimal stage is reached and the optimal assignment is $M_1 \to J_2$, $M_2 \to J_5$, $M_3 \to J_3$, $M_4 \to J_1$, $M_5 \to J_4$,

∴ The minimum machine hours is equal to (6 + 10 + 2 + 4 + 10) *i.e.*, 32 hours.

4. **Find the optimal assignment for the assignment problem with the following cost matrix :**

	W_1	W_2	W_3	W_4
M_1	1	4	6	3
M_2	9	7	10	9
M_3	4	5	11	7
M_4	8	7	8	5

[CP 2014, 2017]

Solution Given assignment cost matrix is a square matrix. So the problem is balanced.

We first subtract least element of each row from all the elements of that row and we get the reduced matrix as given below :

	W_1	W_2	W_3	W_4
M_1	0	3	5	2
M_2	2	0	3	2
M_3	0	1	7	3
M_4	3	2	3	0

Now we subtract least element of each column from all the elements of that column and we get the reduced matrix as below :

	W_1	W_2	W_3	W_4
M_1	$\boxed{0}$	3	2	2
M_2	2	$\boxed{0}$	0	2
M_3	0	1	4	3
M_4	3	2	$\boxed{0}$	0

Now we connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is 3, which is less than the order of the matrix. So the optimal stage is not reached. Now we select '1' which is the least of all surviving elements. We subtract '1' from all the surviving elements and add it to the elements lying in the junction. Then the reduced matrix is given below :

	W_1	W_2	W_3	W_4
M_1	$\boxed{0}$	2	1	1
M_2	3	0	$\boxed{0}$	2
M_3	0	$\boxed{0}$	3	2
M_4	4	2	0	$\boxed{0}$

We again connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is 4, which is equal to the order of the matrix. So, the optimal stage is reached and the optimal assignment is $M_1 \rightarrow W_1$, $M_2 \rightarrow W_3$, $M_3 \rightarrow W_2$, $M_4 \rightarrow W_4$, and the optimal (minimum) cost is $(1 + 10 + 5 + 5)$ *i.e.*, 21 units.

8. **The following is the cost matrix of assigning 4 clerks to 4 key punching jobs. Find the optimal assignment if clerk 1 cannot be assigned to job 1. What is the total minimum cost?**

Job

		1	**2**	**3**	**4**
Clerk	**1**	**—**	**5**	**2**	**0**
	2	**4**	**7**	**5**	**6**
	3	**5**	**8**	**4**	**3**
	4	**3**	**6**	**6**	**2**

Solution In this problem one assignment is not possible. Clark 1 cannot be assigned to the job 1.

To solve this problem we put a very large cost, say 50 (the largest element of the matrix is 8) at the cell (1, 1) to get a new matrix given below and then we solve the problem as usual.

	1	2	3	4
1	50	5	2	0
2	4	7	5	6
3	5	8	4	3
4	3	6	6	2

Since the matrix is square, so the problem is balanced.

We subtract least element of each row from all the elements of that row and get the reduced matrix as below :

	1	2	3	4
1	50	5	2	0
2	0	3	1	2
3	2	5	1	0
4	1	4	4	0

Now we subtract least element of each column from all the elements of that column and get the reduced matrix as below.

	1	2	3	4
1	50	2	1	$\boxed{0}$
2	$\boxed{0}$	0	0	2
3	2	2	$\boxed{0}$	0
4	1	1	3	0

Now we connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is 3, which is less than the order of the matrix. So the optimal stage is not reached. Now we select '1' which is least of all the surviving elements. We subtract '1' from all the surviving elements and add it to the elements lying in the junction. And the reduced matrix is given below :

	1	2	3	4
1	49	1	1	$\boxed{0}$
2	$\boxed{0}$	0	1	3
3	1	1	$\boxed{0}$	0
4	0	$\boxed{0}$	3	0

Case-I

	1	2	3	4
1	49	1	1	$\boxed{0}$
2	0	$\boxed{0}$	1	3
3	1	1	$\boxed{0}$	0
4	$\boxed{0}$	0	3	0

Case-II

We again connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is '4', which is equal to the order of the matrix. So the optimal stage is reached. But in the process of connecting zeros, we see that we can do this in two ways (as shown in the above tables). This situation arises, becuase at one stage of the process, there are two zeros at 2nd and 4th rows and also there are two zeros at 1st and 2nd column. In such conditions alternative solutions exist but the optimal (minimum) cost is same for both solutions.

The optimal solutions are

1 → 4, 2 → 1, 3 → 3, 4 → 2 with minimum cost (0 + 4 + 4 + 6) *i.e.*, 14 units.

and 1 → 4, 2 → 2, 3 → 3, 4 → 1 with minimum cost (0 + 7 + 4 + 3) *i.e.*, 14 units.

10. **Find the minimum cost solution for the assignment problem whose cost coefficients are given in the following table :**

	A	B	C	D
1	–1	1	1	–3
2	2	0	5	–1
3	1	1	–2	3
4	2	3	–1	2

Solution In the table, the most negative number is '–3', so we add '3' to all the elements of the given table and get the reduced table as below :

	A	B	C	D
1	2	4	4	0
2	5	3	8	2
3	4	4	1	6
4	5	6	2	5

The table (matrix) is square, so the assignment problem is balanced.

We first subtract least element of each row from all the elements of that row and get the reduced table as below :

	A	B	C	D
1	2	4	4	0
2	3	1	6	0
3	3	3	0	5
4	3	4	0	3

Now we subtract least element of each column from all the elements of that column and get the reduced table as below :

	A	B	C	D
1	$\boxed{0}$	3	4	0
2	1	$\boxed{0}$	6	0
3	1	2	$\boxed{0}$	5
4	1	3	0	3

We connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is '3', which is less than the order of the matrix. So the optimal stage is not reached. We select '1' which is the least of all the surviving elements. Now we subtract '1' from all the surviving elements and add it to the elements lying in the junction. Then the reduced matrix is given below :

	A	B	C	D
1	0	3	5	$\boxed{0}$
2	1	$\boxed{0}$	7	0
3	$\boxed{0}$	1	0	4
4	0	2	$\boxed{0}$	2

Case-I

	A	B	C	D
1	0	3	5	$\boxed{0}$
2	1	$\boxed{0}$	7	0
3	0	1	$\boxed{0}$	4
4	$\boxed{0}$	2	0	2

Case-II

We again connect all the zeros by minimum number of horizontal and vertical lines. Here the number of lines is 4, which is equal to the order of the matrix. So the optimal stage is reached.

As in the previous example here we get two alternative optimal solutions.

1 → D, 2 → B, 3 → A, 4 → C Cost = –3 + 0 + 1 – 1 = –3 units

and 1 → D, 2 → B, 3 → C, 4 → A Cost = –3 + 0 – 2 + 2 = –3 units.

Game Theory

16. GRAPHICAL METHOD OF SOLUTION.

We illustrate this method by examples.

Consider the game problem whose payoff matrix is given below. The game has no saddle point. Suppose A chooses A_I and A_{II} with probabilities x and $(1 - x)$ respectively. If B chooses B_I along with this, then the expected gain g of A is $g = x + 3(1 - x) = 3 - 2x$.

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{B} \\ & & B_I & B_{II} & B_{III} & B_{IV} \\ \hline A & A_I & 1 & 3 & 0 & 2 \\ \hline & A_{II} & 3 & 0 & 1 & -1 \\ \hline \end{array}$$

If we draw a graph of this function showing g against x, we notice that by choosing B_I throughout, B can restrict A's gain to lie on the line

$$g = 3 - 2x.$$

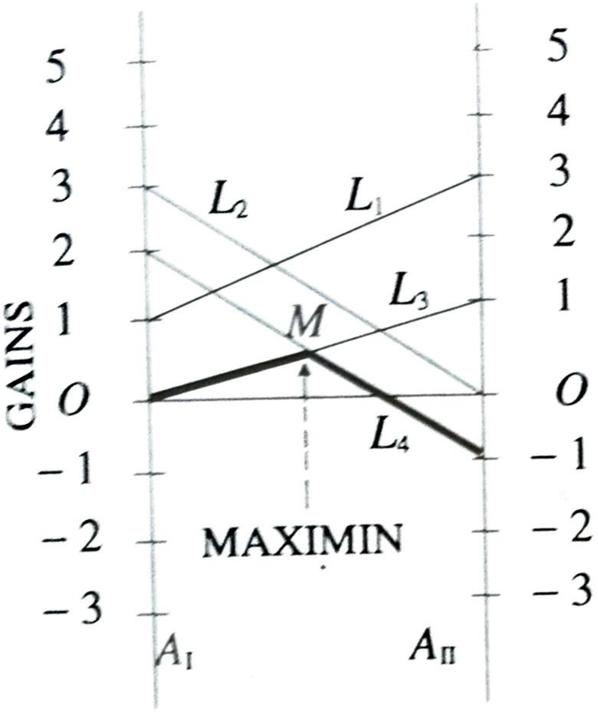
We compute similarly the other gains of A, when B uses B_{II}, B_{III} and B_{IV} and they are

$$g = 3x, \quad g = 1 - x, \quad g = 3x - 1.$$

We draw two parallel axes one unit distance apart and mark a scale on each. The two strategies A_I and A_{II} of A are represented by these two straight lines.

$OO = 1$ unit is the initial line.

The gain lines L_1, L_2, L_3, L_4 are then drawn to represent the gains of A corresponding to B_I, B_{II}, B_{III} and B_{IV} respectively of B by joining



1, 3, 0, 2 on the line representing A_I with 3, 0, 1, (– 1) on the line representing A_{II}.

Now in order to find out the point which will maximize the minimum expected gain of B, we bound the figure so obtained from below by drawing a thick line. The highest point M (maximum) of this bound refers to the maximum of the minimum gains. At this point, as is obvious, B has used his two courses of action and they are B_{III} and B_{IV} as it is the point of intersection of L_3 and L_4.

Therefore the 2×2 sub-matrix which provides the optimal solution of the given game is shown on the right. Now solving as usual this 2×2 sub-game, we get $x = \frac{1}{2}$ and $1 - x = \frac{1}{2}$ (also indicated by the abscissa of the point M).

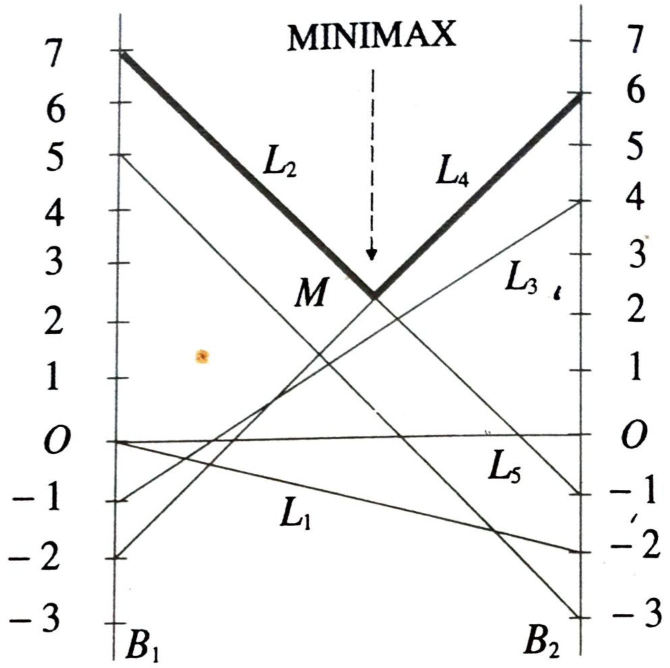
$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{B} \\ & & B_{III} & B_{IV} \\ \hline A & A_I & 0 & 2 \\ \hline & A_{II} & 1 & -1 \\ \hline \end{array}$$

The probabilities for B_{III} and B_{IV} are computed as $\left(\frac{3}{4}, \frac{1}{4}\right)$. The value of the game is $\frac{1}{2}$ (as indicated by the ordinate of the point M).

Thus the optimal strategies are for A $\left(\frac{1}{2}, \frac{1}{2}\right)$ and for B $\left(0, 0, \frac{3}{4}, \frac{1}{4}\right)$ and $v = \frac{1}{2}$.

Consider the game whose payoff matrix is given below along with the graph.

		B	
		B_1	B_2
A	A_1	0	− 2
	A_2	7	− 1
	A_3	− 1	4
	A_4	− 2	6
	A_5	5	− 3



In this graph we have plotted the points of A's pure strategies on the corresponding strategies of B shown by two parallel lines. B_1 and B_2 on the graph at unit distance OO apart. The lowest point M (minimax) of this figure, from above of the thick bound, corresponds to the pair of strategies A_2 and A_4 of A ; hence the 2×2 game which provides the optimal solution of the given game problem is shown on the side and can be solved easily.

		B	
		B_1	B_2
A	A_2	7	− 1
	A_4	− 2	6

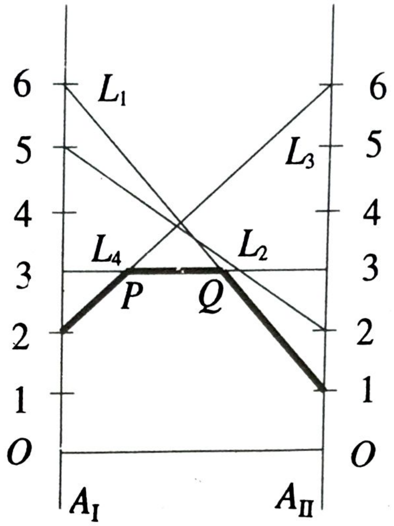
It may sometimes so happen that we get two pairs of strategies having the same value for the game giving two pairs of optimal strategies.

Let us consider the game whose payoff matrix is given below.

		B			
		B_I	B_{II}	B_{III}	B_{IV}
A	A_I	6	5	2	3
	A_{II}	1	2	6	3

Graphical representation of the game as represented by the given matrix is given in the diagram (given in the next page), the manner of computation being the same as in the previous examples. From the graph, it is seen that the player B has two pairs of supporting strategies, for example, (B_I, B_{IV}) and (B_{III}, B_{IV}) giving the same value 3 for the sub-games. The player A has infinite number of mixed optimal strategies (x, $1 - x$), where x

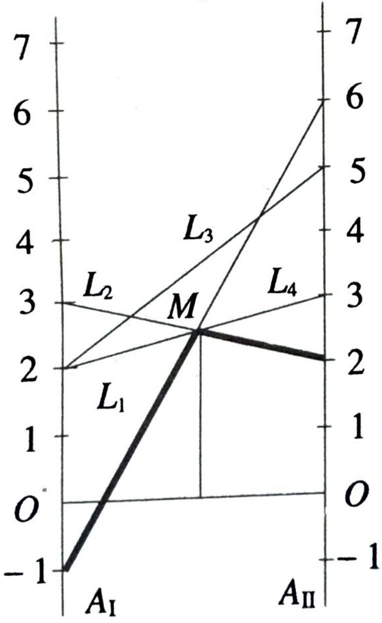
varies from $\frac{2}{5}$ to $\frac{3}{4}$. This can be found either by solving the two 2×2 sub-games or from the abscissae of the points P and Q. It can be shown by solving the two 2×2 sub-games formed by two different pairs of B's strategies that the optimal strategy for B is his pure strategy B_{IV}.



It may so happen in some payoff matrix that three gain lines meet at the minimax or maximin point. In this case to determine the optimal strategy for the players, we solve the 2×2 sub-games corresponding to the gain lines having slopes with opposite sign. The reason for the same will be explained with the graph of such problem.

Consider, for example, the game whose payoff is given by the adjacent matrix. The graphical representation of the game is shown in the diagram given below. It is seen that the three gain lines of A corresponding to B_I, B_{II} and B_{IV} pass through the maximin point M. The value of the game is seen to be $\frac{5}{2}$.

			B		
		B_I	B_{II}	B_{III}	B_{IV}
A	A_I	– 1	3	2	2
	A_{II}	6	2	5	3



Solving the 2×2 sub-game involving B_I and B_{IV} of B, we get the optimal solution as :

for A, optimal strategy is $\left(\frac{1}{2}, \frac{1}{2}\right)$;

for B, optimal strategy is $\left(\frac{1}{8}, \frac{7}{8}, 0, 0\right)$ and the value of the game is $\frac{5}{2}$.

The value of the game is $\frac{5}{2}$ for the 2×2 sub-game with (B_{II} , B_{IV}) also.

But the value of the sub-game with (B_I , B_{IV}) is 3 which is the saddle point of the corresponding sub-matrix. Thus, by choosing (B_I , B_{IV}), B is sure to lose at most

3 while as shown earlier by using (B_{I} , B_{II}) or (B_{II} , B_{IV}), *B* is sure to lose at most 2.5. Thus, for *B*, the mixed strategy using (B_{I} , B_{IV}) will not be optimal.

This is also obvious from the graph. Because, for the two strategies (B_{I} , B_{IV}), the point *M*, the intersection of the lines, is not the maximin point but 3 is the maximin point. But 3 cannot be the value of the original game ; hence the 2 × 2 sub-game with (B_{I} , B_{IV}) will not provide the optimal solution for *B*.

Ex. 3. *Use dominance to reduce the following game problem to 2×2 game and hence find the optimal strategies and the value of the game*

$$\begin{array}{cc} & \text{PLAYER } B \\ \text{PLAYER } A & \begin{array}{|c|c|c|}\hline 3 & -2 & 4 \\ \hline -1 & 4 & 2 \\ \hline 2 & 2 & 6 \\ \hline\end{array}\end{array}$$

[*Visva-Bharati, 1988*]

Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be the probabilities with which A and B play the game.

We see that the first column is dominated by the third column and hence the third column is deleted. The resulting 3×2 payoff matrix is written.

$$\begin{array}{cc} & B \\ A & \begin{array}{|c|c|}\hline 3 & -2 \\ \hline -1 & 4 \\ \hline 2 & 2 \\ \hline\end{array}\end{array}$$

Now we see that the third row dominates a convex linear combination of the first and second row ; for example,

the elements of $\frac{1}{2}$ (first row) + $\frac{1}{2}$ (second row)

are actually $(\frac{3}{2} - \frac{1}{2}), (-1 + 2)$ or, 1, 1

which is dominated by the third row elements 2, 2. Hence either first or second row can be deleted (Rule iv). We delete the first row and the game reduces to the 2×2 payoff matrix

$$\begin{array}{|c|c|}\hline -1 & 4 \\ \hline 2 & 2 \\ \hline\end{array}$$

Solving as in the first example, we get the optimal solution as

$$x_2 = 0, \; x_3 = \frac{-1-4}{-1+2-(4+2)} = \frac{-5}{-5} = 1,$$

$$y_1 = \frac{2-4}{-5} = \frac{2}{5}, \; y_2 = \frac{-1-2}{-5} = \frac{3}{5} \text{ and } v = \frac{-2-8}{-5} = 2.$$

We associate a probability zero with the discarded row and column. The solution of the complete game is thus $A\ (0,\ 0,\ 1)$, $B\left(\frac{2}{5},\ \frac{3}{5},\ 0\right)$ and $v = 2$.

Note. Notice that the final 2×2 matrix may further be reduced, since the second column, which is the dominating column, may be deleted and then from the reduced matrix the first row of that matrix being dominated by the second, may be deleted. Thus the optimal solution becomes for $A\ (0,\ 0,\ 1)$, for $B\ (1,\ 0,\ 0)$ and value of the game is 2.

Ex. 4. *Two players A and B match coins. If the coins match, then A wins one unit of value and if the coins do not match, then B wins one unit of value. Determine the optimum strategies for the players and the value of the game.* [*Delhi M. B. A., 1972*]

The payoff matrix for the matching players A and B in which H and T denote respectively the head and tail of a coin is given below.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{B} \\ & & H & T \\ \hline & H & 1 & -1 \\ \cline{3-4} A & T & -1 & 1 \\ \hline \end{array}$$

As can be seen, there is saddle point for the game and no reduction is possible by dominance. Let $(x_1,\ x_2)$ and $(y_1,\ y_2)$ be the probabilities with which A and B play their pure strategies.

As in the first example, the optimum mixed strategies will be

$$x_1 = \frac{1+1}{1+1-(-1-1)} = \frac{2}{4} = \frac{1}{2}, \quad x_2 = 1 - \frac{1}{2} = \frac{1}{2};$$

$$y_1 = \frac{1-(-1)}{4} = \frac{1}{2}, \quad y_2 = 1 - \frac{1}{2} = \frac{1}{2};$$

$$v = \text{value of the game} = \frac{1-1}{4} = 0.$$

Ex. 5. *For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method :*

$$\begin{array}{cc|c|c|c|} & & \multicolumn{3}{c}{\text{PLAYER } Q} \\ & & Q_1 & Q_2 & Q_3 \\ \cline{3-5} & P_1 & 9 & 1 & 4 \\ \cline{3-5} \text{PLAYER } P & P_2 & 0 & 6 & 3 \\ \cline{3-5} & P_3 & 5 & 2 & 8 \\ \cline{3-5} \end{array}$$

We formulate the problem for the player Q.

$$\text{Maximize } \frac{1}{v} = Y_1 + Y_2 + Y_3,$$

$$\text{subject to } 9Y_1 + Y_2 + 4Y_3 \leq 1,$$

$$6Y_2 + 3Y_3 \leq 1,$$

$$5Y_1 + 2Y_2 + 8Y_3 \leq 1,$$

$$Y_1,\ Y_2,\ Y_3 \geq 0 \text{ and } y_j = v\,Y_j,\ (j = 1,\ 2,\ 3)$$

and v is the maximum expected loss to Q.

We introduce the slack variables Y_4, Y_5, Y_6 respectively and then apply the simplex method. We get the tables by iteration as shown below.

			1	1	1	0	0	0
C_B	Y_B	**b**	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
0	Y_4	1	$\boxed{9}$	1	4	1	0	0
0	Y_5	1	0	6	3	0	1	0
0	Y_6	1	5	2	8	0	0	1
		0	-1	-1	-1	0	0	0
			↑			↓		
1.	Y_1	$\frac{1}{9}$	1	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	0	0
0	Y_5	1	0	$\boxed{6}$	3	0	1	0
0	Y_6	$\frac{4}{9}$	0	$\frac{13}{9}$	$\frac{52}{9}$	$-\frac{5}{9}$	0	1
		$\frac{1}{9}$	0	$-\frac{8}{9}$	$-\frac{5}{9}$	$\frac{1}{9}$	0	0
				↑			↓	
1	Y_1	$\frac{5}{54}$	1	0	$\frac{7}{18}$	$\frac{1}{9}$	$-\frac{1}{54}$	0
1	Y_2	$\frac{1}{6}$	0	1	$\frac{1}{2}$	0	$\frac{1}{6}$	0
0	Y_6	$\frac{11}{54}$	0	0	$\boxed{\frac{91}{18}}$	$-\frac{5}{9}$	$-\frac{13}{54}$	1
		$\frac{7}{27}$	0	0	$-\frac{1}{9}$	$\frac{1}{9}$	$\frac{4}{27}$	0
					↑			↓
1	Y_1	$\frac{21}{273}$	1	0	0	$\frac{14}{91}$	0	$-\frac{7}{91}$
1	Y_2	$\frac{40}{273}$	0	1	0	$\frac{5}{91}$	$\frac{52}{273}$	$-\frac{9}{91}$
1	Y_3	$\frac{11}{273}$	0	0	1	$-\frac{10}{91}$	$-\frac{13}{273}$	$\frac{18}{91}$
		$\frac{24}{91}$	0	0	0	$\frac{9}{91}$	$\frac{13}{91}$	$\frac{2}{91}$

The expected value of the game is thus $\frac{91}{24}$.

Hence $y_1 = v\,Y_1 = \frac{21}{273} \cdot \frac{91}{24} = \frac{7}{24}$.

Similarly $y_2 = \frac{5}{9}$ and $y_3 = \frac{11}{72}$.

The optimum values of X_1, X_2, X_3 are obtained from the dual solution $\frac{9}{91}, \frac{13}{91}, \frac{2}{91}$.

Hence $x_1 = vX_1 = \frac{9}{91} \cdot \frac{91}{24} = \frac{3}{8}$.

Similarly $x_2 = \frac{13}{24}$ and $x_3 = \frac{1}{12}$.

Hence the optimum solution is given by $P\left(\frac{3}{8}, \frac{13}{24}, \frac{1}{12}\right)$, $Q\left(\frac{7}{24}, \frac{5}{9}, \frac{11}{72}\right)$ and $v = \frac{91}{24}$.