3. SETS IN R

3.1. Intervals.

Let $a, b \in \mathbb{R}$ and $a < b$.

The subset $\{x \in \mathbb{R} : a < x < b\}$ is said to be an *open interval.* The points a and b are called the *end points* of the interval. a and b are not points in the open interval. This open interval is denoted by (a, b).

The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is said to be a *closed interval.* The end points a and b are points in the closed interval. This closed interval is denoted by $[a, b]$.

The subsets $\{x \in \mathbb{R} : a < x \leq b\}$ and $\{x \in \mathbb{R} : a \leq x < b\}$ are said to be *half open* (or *half closed*) intervals. One of the end points is a point in the interval. These half open intervals are denoted by $(a, b]$ and $[a, b)$ respectively.

The subset $\{x \in \mathbb{R} : x > a\}$ is an *infinite open interval.* This is denoted by (a, ∞).

The subset $\{x \in \mathbb{R} : x \geq a\}$ is an *infinite closed interval.* This is denoted by $[a, \infty)$

The subset $\{x \in \mathbb{R} : x < a\}$ is an *infinite open interval.* This is denoted by $(-\infty, a)$.

The subset $\{x \in \mathbb{R} : x \leq a\}$ is an *infinite closed interval.* This is denoted by $(-\infty, a]$.

When both the end points of an interval belong to $\mathbb{R}$, the interval is said to be a *bounded interval.*

Therefore the intervals (a, b), $[a, b]$, $(a, b]$, $[a, b)$ are all bounded intervals.

The intervals (a, ∞), $[a, \infty)$, $(-\infty, a)$, $(-\infty, a]$ are all *unbounded intervals.*

If $a = b$, the closed interval $[a, a]$ is the singleton set $\{a\}$.

The set $\mathbb{R}$ is also denoted by $(-\infty, \infty)$. This is an unbounded interval without end points.

3.2. Neighbourhood.

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a *neighbourhood* of c if there exists an open interval (a, b) such that $c \in (a, b) \subset S$.

Clearly, an open bounded interval containing the point c is a neighbourhood of c. Such a neighbourhood of c is denoted by $N(c)$.

A closed bounded interval containing the point c may not be a neighbourhood of c. For example, $1 \in [1, 3]$ but $[1, 3]$ is not a neighbourhood of 1.

Let $c \in \mathbb{R}$ and $\delta > 0$. The open interval $(c - \delta, c + \delta)$ is said to the δ-neighbourhood of c and is denoted by $N(c, \delta)$. Clearly, the δ-neighbourhood of c is an open interval symmetric about c.

Theorem 3.2.1. Let $c \in \mathbb{R}$. The union of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$. Let $a_3 = \min\{a_1, a_2\}$, $b_3 = \max\{b_1, b_2\}$. Then $(a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

$(a_1, b_1) \subset S_1 \cup S_2$ and $(a_2, b_2) \subset S_1 \cup S_2 \Rightarrow (a_3, b_3) = (a_1, b_1) \cup (a_2, b_2) \subset S_1 \cup S_2$.

Thus $c \in (a_3, b_3) \subset S_1 \cup S_2$.

This proves that $S_1 \cup S_2$ is a neighbourhood of c.

Note. The union of a finite number of neighbourhoods of c is a neighbourhood of c.

Theorem 3.2.2. Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$. Let $a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$. Then $(a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

$(a_3, b_3) = (a_1, b_1) \cap (a_2, b_2) \subset (a_1, b_1) \subset S_1$ and $(a_3, b_3) = (a_1, b_1) \cap (a_2, b_2) \subset (a_2, b_2) \subset S_2 \Rightarrow (a_3, b_3) \subset S_1 \cap S_2$.

Thus $c \in (a_3, b_3) \subset S_1 \cap S_2$.

This proves that $S_1 \cap S_2$ is a neighbourhood of c.

Note. The intersection of a finite number of neighbourhoods of a point c is a neighbourhood of c.

The intersection of an infinite number of neighbourhoods of a point c may not be a neighbourhood of c.

For example, for every $n \in \mathbb{N}$, $(-\frac{1}{n}, \frac{1}{n})$ is a neighbouhood of 0. $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$. This is not a neighbourhood of 0.

3.3. Interior point.

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an *interior point* of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

The set of all interior points of S is said to be the *interior* of S and is denoted by int S (or by S^o).

From definition it follows that $S^o \subset S$ for any set $S \subset \mathbb{R}$.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots \cdots\}$.

Let $x \in S$. Every neighbourhood of x contains some points not in S. So x can not be an interior point of S. Therefore int $S = \phi$.

2. Let $S = \mathbb{N}$.

Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So x can not be an interior point of S. Therefore int $S = \phi$.

3. Let $S = \mathbb{Q}$.

Let $x \in \mathbb{Q}$. Every neighbourhood of x contains rational as well as irrational points. So x can not be an interior point of $\mathbb{Q}$. So $S^o = \phi$.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is an interior point of S. So int $S = S$.

5. Let $S = \mathbb{R}$. Each point of S is an interior point of S. Therefore $S^o = S$.

6. Let $S = \phi$. S has no interior point. Therefore int $S = \phi$.

3.4. Open set.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be an *open set* if each point of S is an interior point of S.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$. No point of S is an interior point of S. S is not an open set.

2. Let $S = \mathbb{Z}$. No point of S is an interior point of S. S is not an open set.

3. Let $S = \mathbb{Q}$. No point of S is an interior point of S. S is not an open set.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is an interior point of S. S is an open set.

5. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. 1 and 3 belong to S but they are not interior points of S. S is not an open set.

6. Let $S = \mathbb{R}$. Each point of S is an interior point of S. S is an open set.

7. Let $S = \phi$. S contains no point. Therefore the requirement in the definition is vacuously satisfied. S is an open set.

Theorem 3.4.1. Let $S \subset \mathbb{R}$. Then S is an open set if and only if $S =$ int S.

Proof. We prove the theorem for a non-empty set S because if $S = \phi$ then $\phi =$ int ϕ holds and also ϕ is an open set.

Let S be a non-empty open set and let $x \in S$. Then x is an interior point of S.

Thus $x \in S \Rightarrow x \in$ int S. Therefore $S \subset$ int S ... ... (i)

Let $y \in$ int S. Then $y \in S$ by the definition of an interior point.

Thus $y \in$ int $S \Rightarrow y \in S$. Therefore int $S \subset S$ ... ... (ii)

From (i) and (ii) we have $S =$ int S.

Conversely, let S be a non-empty set and $S =$ int S.

Let $x \in S$. Then $x \in$ int S, since $S =$ int S.

Thus each point of S is an interior point of S and therefore S is an open set.

This completes the proof.

Theorem 3.4.2. The union of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Let $x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$.

Let $x \in G_1$. Since G_1 is open set and $x \in G_1, x$ is an interior point of G_1. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_1$.

$N(x) \subset G_1 \Rightarrow N(x) \subset G_1 \cup G_2$.

This shows that x is an interior point of $G_1 \cup G_2$.

Since x is arbitrary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is an open set.

If however, $x \in G_2$, we can prove in a similar manner that $G_1 \cup G_2$ is an open set. This completes the proof.

Theorem 3.4.3. The intersection of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Case 1. $G_1 \cap G_2 = \phi$. Since ϕ is an open set, $G_1 \cap G_2$ is an open set.

Case 2. $G_1 \cap G_2 \neq \phi$. Let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$.

Since G_1 is an open set and $x \in G_1, x$ is an interior point of G_1.

Hence there exists a positive δ_1 such that the neighbourhood $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2, x$ is an interior point of G_2.

Hence there exists a positive δ_2 such that the neighbourhood $N(x, \delta_2) \subset G_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$.

$N(x, \delta) \subset N(x, \delta_1) \subset G_1$ and $N(x, \delta) \subset N(x, \delta_2) \subset G_2$.

Consequently, $N(x, \delta) \subset G_1 \cap G_2$.

This shows that x is an interior point of $G_1 \cap G_2$. Since x is arbitrary, $G_1 \cap G_2$ is an open set and this completes the proof.

Theorem 3.4.4. The union of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof. Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cup G_2 \cup \cdots \cup G_m$.

Let $x \in G$. Then x belongs to at least one of the sets, say G_k. Since G_k is an open set and $x \in G_k, x$ is an interior point of G_k. Hence there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_k$.

$N(x) \subset G_k \Rightarrow N(x) \subset G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set. This completes the proof.

Theorem 3.4.5. The intersection of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof. Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_m$.

Case 1. $G = \phi$. Then G is an open set, since ϕ is an open set.

Case 2. $G \neq \phi$. Let $x \in G$. Then $x \in G_i$ for each $i = 1, 2, \ldots, m$.

Since G_1 is an open set and $x \in G_1$, there exists a positive δ_1 such that $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2$, there exists a positive δ_2 such that $N(x, \delta_2) \subset G_2$.

... ...

Since G_m is an open set and $x \in G_m$, there exists a positive δ_m suc
that $N(x, \delta_m) \subset G_m$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$.

$N(x,\delta) \subset N(x,\delta_1) \subset G_1$

$N(x,\delta) \subset N(x,\delta_2) \subset G_2$

$\ldots \quad \ldots \quad \ldots$

$N(x,\delta) \subset N(x,\delta_m) \subset G_m$.

Consequently, $N(x,\delta) \subset G_1 \cap G_2 \cap \cdots \cap G_m = G$.

This shows that x is an interior point of G. Since x is arbitrary, G i
an open set. This completes the proof.

Theorem 3.4.6. The union of an arbitrary collection of open sets in
is an open set.

Proof. Let $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be an arbitrary collectio
of open sets in $\mathbb{R}$. Let $G = \underset{\alpha \in \Lambda}{\cup} G_\alpha$.

Let $x \in G$. Then x belongs to at least one open set of the collectio
say $G_\lambda, (\lambda \in \Lambda)$.

Since G_λ is an open set and $x \in G_\lambda$, x is an interior point of G_λ.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x)$
G_λ. $N(x) \subset G_\lambda \Rightarrow N(x) \subset G$.

This shows that x is an interior point of G. Since x is arbitrary, G i
an open set and the proof is complete.

Note. The intersection of an infinite number of open sets in $\mathbb{R}$ is no
necessarily an open set.

Let us consider the sets G_i where

$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$

$G_2 = \{x \in \mathbb{R} : -\frac{1}{2} < x < \frac{1}{2}\}$

$\ldots \quad \ldots \quad \ldots$

$G_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}\}$

$\ldots \quad \ldots \quad \ldots$

Each G_i is an open set. $\bigcap_{i=1}^{\infty} G_i = \{0\}$. This is not an open set.

Let us consider the sets G_i where

$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$

$G_2 = \{x \in \mathbb{R} : -2 < x < 2\}$

$\ldots \quad \ldots \quad \ldots$

$G_n = \{x \in \mathbb{R} : -n < x < n\}$

$\ldots \quad \ldots \quad \ldots$

Each G_i is an open set. $\bigcap_{i=1}^{\infty} G_i = G_1$. This is an open set.

From these two examples we conclude that the intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Theorem 3.4.7. Let S be a subset of $\mathbb{R}$. Then int S is an open set.

Proof. **Case 1.** int $S = \phi$. Since ϕ is an open set, int S is an open set.

Case 2. int $S \neq \phi$. Let $x \in$ int S. Then x is an interior point of S. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

Let $y \in N(x)$. Then $N(x)$ is a neighbourhood of y also and since $N(x) \subset S$, y is an interior point of S.

Thus $y \in N(x) \Rightarrow y \in$ int S. Therefore $N(x) \subset$ int S.

This shows that x is an interior point of int S.

Thus $x \in$ int $S \Rightarrow x$ is an interior point of int S.

Therefore int S is an open set. This completes the proof.

Theorem 3.4.8. Let $S \subset \mathbb{R}$. Then int S is the largest open set contained in S.

Proof. By the previous theorem, int S is an open set and int $S \subset S$, by definition.

Let P be any open set contained in S.

Let $x \in P$. Since P is an open set, x is an interior point of P.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset P$. But $N(x) \subset P \Rightarrow N(x) \subset S$, since $P \subset S$.

This shows that x is an interior point of S, i.e. $x \in$ int S.

Thus $x \in P \Rightarrow x \in$ int S. Therefore $P \subset$ int S.

Since P is arbitrary, int S is the largest open set contained in S.

Note. int S is the union of all open sets contained in S.

Worked Examples.

1. Prove that an open interval is an open set.

Let I be an open interval. Four cases arise.

Case 1. $I = (a, b)$ for some $a, b \in \mathbb{R}$, with $a < b$.

Let $c \in I$. Then I itself is a neighbourhood of c, say $N(c)$ and $N(c) \subset I$. This shows that c is an interior point of I. Thus every point of I is an interior point of I and therefore I is an open set.

Case 2. $I = (a, \infty)$ for some $a \in \mathbb{R}$.

Let $c \in I$. Then $a < c < \infty$. Let $d \in (c, \infty)$. Then $a < c < d$.

The open interval (a, d) is a neighbourhood of c, say $N(c)$ and $N(c) \subset I$. This shows that c is an interior point of I. Thus every point of I is an interior point of I and therefore I is an open set.

Case 3. $I = (-\infty, a)$ for some $a \in \mathbb{R}$.
Similar proof.

Case 4. $I = (-\infty, \infty)$.
Similar proof.

2. Let $S = (0, 1]$ and $T = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$. Show that $S - T$ is an open set.

$S - T = (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup \cdots$

$S - T$ is the union of an infinite number of open intervals. Since an open interval is an open set, $S - T$ being the union of an infinite number of open sets is an open set.

We have seen that an open interval is an open set in $\mathbb{R}$ and the union of any collection of open sets is an open set in $\mathbb{R}$. Therefore the union of an arbitrary collection of open intervals is an open set in $\mathbb{R}$.

The following theorem deals with the converse problem and it depicts the structural composition of a bounded open set in $\mathbb{R}$.

Theorem 3.4.9. A non-empty bounded open set in $\mathbb{R}$ is the union of countable collection of disjoint open intervals.

Proof. Let G be a non-empty bounded open set in $\mathbb{R}$. Let $x \in G$. Since G is an open set, there is an element $y_o < x$ and an element $z_o > x$ such that $(y_o, x) \subset G$ and $(x, z_o) \subset G$.

Let $A = \{y : (y, x) \subset G\}$, $B = \{z : (x, z) \subset G\}$.

Then A is a non-empty set, since $y_o \in A$; A is bounded below, since G is bounded below. Let $a = \inf A$.

Similarly, B is a non-empty set bounded above. Let $b = \sup B$.

Then $a < x < b$ and $I_x = (a, b)$ is an open interval containing x. We prove $I_x \subset G$.

Let $w \in I_x$ and $a < x < w < b$.

Since $b = \sup B$, there exists an element $z' \in B$ such that $w < z' \le b$.

Therefore $(x, z') \subset G$, since $z' \in B$. Therefore $w \in G$.

If however, $w \in I_x$ and $a < w < x < b$, then also $w \in G$.

Thus $w \in I_x \Rightarrow w \in G$ and therefore $I_x \subset G$.

We prove $a \notin G, b \notin G$.

If $b \in G$, then for some positive ϵ, $(b - \epsilon, b + \epsilon) \subset G$, since G is an open set. Let $\delta < \epsilon$. Then $b < b + \delta < b + \epsilon$ and $b + \delta \in G$ contradicting the definition of b. Therefore $b \notin G$. Similarly, $a \notin G$.

Let $\mathcal{G}$ be the collection of open intervals $\{I_x : x \in G\}$. Let $H = \bigcup_{x \in G} I_x$

Let $x \in G$. Then $x \in I_x$ and $I_x \subset H$.

Thus $x \in G \Rightarrow x \in H$. Therefore $G \subset H$.

Let $y \in H$. Then $y \in I_y$ and $I_y \subset G$.

Thus $y \in H \Rightarrow y \in G$. Therefore $H \subset G$.

Consequently, $G = H = \bigcup_{x \in G} x$.

We prove that two distinct intervals in the collection $\mathcal{G}$ are disjoint.

Let $(a, b), (c, d)$ be two intervals in this collection with a point p in common.

Then $c < b$ and $a < d$.

Since $c \notin G$, c does not belong to (a, b) and therefore $c \le a$.

Since $a \notin G$, a does not belong to (c, b) and therefore $a \le c$.

$c \le a$ and $a \le c \Rightarrow a = c$. Similarly $b = d$.

Therefore two distinct intervals of the collection are disjoint.

Thus G is the union of disjoint collection of open intervals $\{I_x : x \in G\}$.

We show that the collection is countable.

Let $\mathcal{G}'$ be the collection $\{I_\alpha : \alpha \in \Lambda\}$ where I_α is an open interval and Λ is the index set.

Let $\lambda \in \Lambda$. Then I_λ is an open interval of the collection $\mathcal{G}'$.

Let $x \in I_\lambda$. Then there exists a positive δ such that $(x - \delta, x + \delta) \subset I_\lambda$. There exists a rational number r_λ such that $x - \delta < r_\lambda < x + \delta$. Therefore $r_\lambda \in \mathbb{Q} \cap I_\lambda$.

Let us define a function $f : \Lambda \to \mathbb{Q}$ that assigns $\lambda (\in \Lambda)$ to $r_\lambda (\in \mathbb{Q})$.

Since I_α's are disjoint, the function f is injective.

Since $\mathbb{Q}$ is an enumerable set and f is an injective function, Λ is at most enumerable. Hence $\mathcal{G}'$ is a countable collection.

This completes the proof.

3.5. Limit point.

Definition. Let S be a subset of $\mathbb{R}$. A point p in $\mathbb{R}$ is said to be a *limit point* (or an *accumulation point*, or a *cluster point*) of S if every neighbourhood of p contains a point of S other than p.

Therefore p is a limit point of S if for each positive ϵ,

$$[N(p, \epsilon) - \{p\}] \cap S \neq \phi.$$

$N(p, \epsilon) - \{p\}$ is called the *deleted* ϵ-neighbourhood of p and is denoted by $N'(p, \epsilon)$. $N(p) - \{p\}$ is called the *deleted neighbourhood* of p and is denoted by $N'(p)$.

Therefore p is a limit point of S if every deleted neighbourhood of p contains a point of S.

Theorem 3.6.2. Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ both exist. Let $s^* = \sup S$ and $s_* = \inf S$. Then $x \in S \Rightarrow s_* \leq x \leq s^*$.

Let H be a subset of $\mathbb{R}$ defined by $H = \{x \in \mathbb{R} : x$ is greater than infinitely many elements of $S\}$.

Then $s^* \in H$ and so H is a non-empty subset of $\mathbb{R}$.

Let $h \in H$. Then h is greater than infinitely many elements of S and therefore $h > s_*$, because no element $\leq s_*$ exceeds infinitely many elements of S.

Thus H is a non-empty subset of $\mathbb{R}$, bounded below, s_* being a lower bound. So $\inf H$ exists.

Let $\inf H = \xi$. We now show that ξ is a limit point of S.

Let us choose $\epsilon > 0$.

Since $\inf H = \xi$, there exists an element y in H such that $\xi \leq y < \xi + \epsilon$.

Since $y \in H$, y exceeds infinitely many elements of S and consequently $\xi + \epsilon$ exceeds infinitely many element of S.

Since ξ is the infimum of H, $\xi - \epsilon$ does not belong to H and so $\xi - \epsilon$ can exceed at most a finite number of elements of S. Thus the neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ contains infinitely many elements of S.

This holds for each $\epsilon > 0$. Therefore ξ is a limit point of S.

This completes the proof.

3.7. Derived set.

Definition. Let S be a subset of $\mathbb{R}$. The set of all limit points of S is said to be the *derived set* of S and is denoted by S'.

Examples.

1. Let S be a finite set. Then $S' = \phi$.
2. Let $S = \mathbb{N}$. Then $S' = \phi$.
3. Let $S = \mathbb{Z}$. Then $S' = \phi$.
4. Let $S = \mathbb{Q}$. Then $S' = \mathbb{R}$.
5. Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$.
6. Let $S = \phi$. Then $S' = \phi$.

Theorem 3.7.1. Let A, B be subsets of $\mathbb{R}$ and $A \subset B$. Then $A' \subset B'$.

Proof. **Case 1.** $A' = \phi$. Then $A' \subset B'$.

Case 2. $A' \neq \phi$. Let $p \in A'$. Then p is a limit point point of A.

Let $\epsilon > 0$. Then $N(p, \epsilon)$ contains a point of A, say q, other than p. $q \in A \Rightarrow q \in B$. Therefore $N'(p, \epsilon)$ contains a point q of B.

Since ϵ is arbitrary, p is a limit point of B. Therefore $p \in B'$. Thus $p \in A' \Rightarrow p \in B'$ and therefore $A' \subset B'$.

This completes the proof.

Theorem 3.7.2. Let $A \subset \mathbb{R}$. Then $(A')' \subset A'$.

Proof. **Case 1.** $(A')' = \phi$. Then $(A')' \subset A'$.

Case 2 . $(A')' \neq \phi$. Let $p \in (A')'$. Then p is a limit point of A'.

Let $\epsilon > 0$. Then $N(p, \epsilon)$ contains a point of A', say q, other than p.

Since $q \in A'$, q is a limit point of A. Therefore $N(p, \epsilon)$ being a neighbourhood of q also, contains infinitely many points of A.

Since $N(p, \epsilon)$ contains infinitely many points of A, p is a limit point of A. That is, $p \in A'$.

Thus $p \in (A')' \Rightarrow p \in A'$ and therefore $(A')' \subset A'$.

This completes the proof.

Theorem 3.7.3. Let $A, B \subset \mathbb{R}$. Then $(A \cap B)' \subset A' \cap B'$.

Proof. $A \cap B \subset A \Rightarrow (A \cap B)' \subset A'$, since $A \subset B \Rightarrow A' \subset B'$.

$A \cap B \subset B \Rightarrow (A \cap B)' \subset B'$, since $A \subset B \Rightarrow A' \subset B'$.

It follows that $(A \cap B)' \subset A' \cap B'$.

Note. $(A \cap B)' \neq A' \cap B'$, in general.

For example, let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, $B = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \cdots\}$. Then $A' = \{0\}$, $B' = \{0\}$. $A \cap B = \{0\}$, $A' \cap B' = \{0\}$, but $(A \cap B)' = \phi$.

Corollary. Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cap A_2 \cap \cdots \cap A_m)' \subset A_1' \cap A_2' \cap \cdots \cap A_m'$.

Theorem 3.7.4. Let A and B be subsets of $\mathbb{R}$. Then $(A \cup B)' = A' \cup B'$.

Proof. $A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$, since $A \subset B \Rightarrow A' \subset B'$

$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$, since $A \subset B \Rightarrow A' \subset B'$.

It follows that $A' \cup B' \subset (A \cup B)'$ ... ... (i)

We now prove that $(A \cup B)' \subset A' \cup B'$.

Let $p \notin A' \cup B'$. Then $p \notin A'$ and $p \notin B'$.

10. Let $G \subset \mathbb{R}$ be an open set and $F \subset \mathbb{R}$ be a closed set. Prove that $G - F$ is an open set and $F - G$ is a closed set.

11. Let G be an open set in $\mathbb{R}$ and S be a non-empty finite subset of G. Prove that $G - S$ is an open set.

12. Let G be an open set in $\mathbb{R}$ and S be a subset of $\mathbb{R}$ such that $G \cap S = \phi$. Prove that $G \cap S' = \phi$.

[**Hint.** $G \cap S = \phi \Rightarrow S \subset G^c \Rightarrow S' \subset (G^c)' \Rightarrow S' \subset G^c$(since G^c is closed in $\mathbb{R}$)$\Rightarrow G \cap S' = \phi$.]

13. G is an open set in $\mathbb{R}$ and $A \subset \mathbb{R}$. Prove that $G \cap \bar{A} \subset \overline{G \cap A}$. Deduce that $\overline{G \cap \bar{A}} = \overline{G \cap A}$.

[**Hint.** (i) $G \cap A \subset \overline{G \cap A}$ and $G \cap A' \subset \overline{G \cap A}$. (ii)$\overline{G \cap \bar{A}}$ being the smallest closed set containing $G \cap \bar{A}$, $\overline{G \cap \bar{A}} \subset \overline{G \cap A}$; and $\overline{G \cap A} \subset \overline{G \cap A} \cup \overline{G \cap A'}$.]

14. Let S be a bounded subset of $\mathbb{R}$ and $\sup S = b, \inf S = a$ and $a \neq b$. Prove that $[a, b]$ is the smallest closed interval containing the set S.

15. Let S be a non-empty subset of $\mathbb{R}$ bounded below and $S_* = \inf S$. If $S_* \notin S$, prove that S_* is a limit point of S and S_* is the least element of S'.

16. If S be a non-empty bounded subset of $\mathbb{R}$, prove that $\sup S \in \bar{S}$ and $\inf S \in \bar{S}$.

17. (i) Prove that ext $(A \cup B) =$ (ext A)$\cap$ (ext$B$) for subsets A, B of $\mathbb{R}$.

(ii) Prove that the complement of the exterior of a subset S of $\mathbb{R}$ is the closure of S.

(iii) Prove that the exterior of the complement of a subset S of $\mathbb{R}$ is the interior of S.

18. A set $S \subset \mathbb{R}$ is said to be a *discrete* set if $S' = \phi$.

A set $S \subset \mathbb{R}$ is said to be an *isolated* set if $S \cap S' = \phi$ (i.e., if each point of S is an isolated point).

(i) Prove that every discrete set is an isolated set, but not conversely.

(ii) Give an example of an infinite discrete set $S \subset \mathbb{R}$.

(iii) Give an example of a bounded discrete set $S \subset \mathbb{R}$.

(iv) Can there be an infinite bounded discrete set $S \subset \mathbb{R}$?

19. Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a *boundary point* of S if every neighbourhood $N(x)$ of x contains a point of S and also a point of $\mathbb{R} - S$.

If a boundary point of S is not a point of S, prove that it is a limit point of S. Prove that a set $S \subset \mathbb{R}$ is closed if and only if S contains all it boundary points.

3.11. Nested intervals.

If $\{I_n : n \in \mathbb{N}\}$ be a family of intervals such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then the family $\{I_n\}$ is said to be a family of *nested intervals*.

Examples.

1. Let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$.
   Then $I_1 = (0, 1)$, $I_2 = (0, \frac{1}{2})$, $I_3 = (0, \frac{1}{3}), \cdots \cdots$
   $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
   $\{I_n : n \in \mathbb{N}\}$ is a family of nested open and bounded intervals.

2. Let $I_n = \{x \in \mathbb{R} : x > n\}$.
   Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
   $\{I_n : n \in \mathbb{N}\}$ is a family of nested open infinite intervals.

3. Let $I_n = \{x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\}$.
   Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
   $\{I_n : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals.

4. Let $I_n = \{x \in \mathbb{R} : x \leq \frac{1}{n}\}$.
   Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
   $\{I_n : n \in \mathbb{N}\}$ is a family of nested closed infinite intervals.

Theorem 3.11.1. Theorem on nested intervals.

If $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a family of nested closed and bounded intervals then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Furthermore, if $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, then there is *one and only one point* x such that $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Proof. $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \cdots$

Then $a_1 \leq a_2 \leq a_3 \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_3 \leq b_2 \leq b_1$.

The set $A = \{a_i : i \in \mathbb{N}\}$ is a non-empty subset of $\mathbb{R}$ bounded above, b_1 being an upper bound. By the supremum property of $\mathbb{R}$, $\sup A$ exists. Let $\sup A = x$. Then $a_n \leq x$ for all $n \in \mathbb{N}$.

We now establish that $b_n \geq x$ for all $n \in \mathbb{N}$.

If not, let $b_m < x$ for some $m \in \mathbb{N}$.

Since x is the lub of the set $\{a_1, a_2, a_3, \ldots\}$ and $b_m < x$, there is an element a_k such that $b_m < a_k < x$.

Let $q = \max\{m, k\}$. Then $b_q \leq b_m$ and $a_k \leq a_q$.

Consequently, $b_q \leq b_m < a_k \leq a_q$.

This shows that $b_q < a_q$, a contradiction, since $[a_q, b_q]$ is an interval of the family.

Hence $b_n \geq x$ for all $n \in \mathbb{N}$ and therefore $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$.
That is, $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

This proves that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Second part. If possible, let $x' \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Then $a_n \leq x \leq b_n$, $a_n \leq x' \leq b_n$ for all $n \in \mathbb{N}$.
Therefore $a_n - b_n \leq x - x' \leq b_n - a_n$.
or, $0 \leq | x - x' | \leq b_n - a_n$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0, b_n - a_n \geq 0$ for all $n \in \mathbb{N}$ and there exists an element $b_m - a_m$ of the set (corresponding to some natural number m) such that $0 \leq b_m - a_m < \epsilon$.

Therefore $0 \leq | x - x' | < \epsilon$. Since ϵ is arbitrary, $x = x'$.

This proves that x is unique and the proof is complete.

Note 1. The set $B = \{b_i : i \in \mathbb{N}\}$ is a non-empty subset of $\mathbb{R}$ bounded below. If $\inf B = y$, then $y \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Note 2. If $\{I_n : n \in \mathbb{N}\}$ be a family of nested open bounded intervals then $\bigcap_{n=1}^{\infty} I_n$ may not be non-empty.

For example, if $I_n = (0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} I_n = \phi$.

Note 3. If $\{I_n : n \in \mathbb{N}\}$ be a family of nested closed unbounded intervals then $\bigcap_{n=1}^{\infty} I_n$ may not be non-empty.

For example, if $I_n = [n, \infty)$, then $\bigcap_{n=1}^{\infty} I_n = \phi$.

Utilising Nested intervals theorem we now give an alternative proof of Bolzano-Weierstrass theorem (Theorem 3.6.2).

Another proof of Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof. Let S be a bounded subset of $\mathbb{R}$ containing infinite number of elements. Since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ exist. Let $a_1 = \inf S$, $b_1 = \sup S$.

Then $x \in S \Rightarrow a_1 \leq x \leq b_1$, i.e., $x \in [a_1, b_1]$. Thus S is contained in the closed and bounded interval $I_1 = [a_1, b_1]$.

Let $c_1 = \frac{a_1+b_1}{2}$. Then at least one of the closed intervals $[a_1, c_1], [c_1, b_1]$ must contain infinitely many elements of S. Because, otherwise, S would be a finite set. We take one such subinterval containing infinitely many elements of S and call it $I_2 = [a_2, b_2]$.

$I_2 \subset I_1$ and $| I_2 | = \frac{b_1 - a_1}{2}$.

Let $c_2 = \frac{a_2+b_2}{2}$. Then at least one of the closed intervals $[a_2, c_2], [c_2, b_2]$ must contain infinitely many elements of S. We take one such subinterval containing infinitely many elements of S and call it $I_3 = [a_3, b_3]$.

$I_3 \subset I_2 \subset I_1$ and $| I_3 | = \frac{b_1 - a_1}{2^2}$.

Let $c_3 = \frac{a_3+b_3}{2}$. Continuing in a similar manner we obtain a family of closed and bounded intervals $\{I_n\}$ such that

(i) $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$

(ii) $| I_n | = \frac{1}{2^{n-1}}(b_1 - a_1)$, for each $n \in \mathbb{N}$

(iii) I_n contains infinitely many elements of S, for each $n \in \mathbb{N}$.

So $\{I_n : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals and $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$.

By the nested intervals theorem, there exists precisely one point x such that $\{x\} = \bigcap_{n=1}^{\infty} [a_n, b_n]$.

We now prove that x is a limit point of S.

Let $\epsilon > 0$. Since $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, there exists a natural number m such that $0 \leq b_m - a_m < \epsilon$.

Since $x \in I_m$ and $b_m - a_m < \epsilon, I_m \subset N(x, \epsilon)$.

Since I_m contains infinitely many elements of $S, N(x, \epsilon)$ contains infinitely many elements of S and this happens for each $\epsilon > 0$.

Therefore x is a limit point of S.

Thus S has a limit point and the theorem is done.

Theorem 3.11.2. Cantor's intersection theorem.

Let $F_1, F_2, F_3, \ldots \ldots$ be a countable collection of non-empty closed and bounded subsets of $\mathbb{R}$ such that $F_1 \supset F_2 \supset F_3 \supset \cdots \cdots$

Then the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.

Proof. **Case 1.** Let the collection be a finite collection containing m non-empty closed and bounded subsets $F_1, F_2, \ldots, F_m$ such that $F_1 \supset F_2 \supset \cdots \supset F_m$.

Then obviously, $\bigcap_{i=1}^{m} F_i = F_m$ and this is non-empty by hypothesis.

Case 2. Let the collection be countably infinite. Without loss of generality, we we assume that no two sets of the collection are equal sets.

4. REAL FUNCTIONS

4.1. Real function.

Let X be a non-empty set. A function $f : X \to \mathbb{R}$ is called a *real valued function* on X. For each $x \in X$, the f-image, denoted by $f(x)$ (which is also called the value of f at x), is a real number.

For example, the function $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = |z|, z \in \mathbb{C}$ is a real valued function of complex numbers.

Let D be a non-empty subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be a *real valued function of real numbers*. Such a function is also called a *real function*.

D is said to be the *domain* of f. The set $f(D) = \{f(x) : x \in D\}$ is a subset of $\mathbb{R}$ and it is called the *range* of f.

Examples.

1. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$, $x \in \mathbb{R}$. The range of the function f is the singleton set $\{c\}$. f is called a **constant function**.

2. Let $D = \{x \in \mathbb{R} : x \neq 0\}$ and $f : D \to \mathbb{R}$ is defined by $f(x) = \frac{1}{x}, x \neq 0$. The range of f is $\{x \in \mathbb{R} : x \neq 0\}$.

3. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f : D \to \mathbb{R}$ is defined by $f(x) = \sqrt{x}, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 0\}$. f is called the **square root function**.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined $f(x) = \sin x, x \in \mathbb{R}$. The range of f is $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$. f is called the real **sine function**.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = | x |, x \in \mathbb{R}$. The range of the function is $\{x \in \mathbb{R} : x \geq 0\}$. f is equivalently expressed as

$$\begin{aligned} f(x) &= x, x > 0 \\ &= 0, x = 0 \\ &= -x, x < 0. \end{aligned}$$

f is called the **absolute value function.**

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \text{sgn } x, x \in \mathbb{R}$.

$$\begin{aligned} \text{sgn } x &= \tfrac{|x|}{x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

The range of f is the finite set $\{-1, 0, 1\}$. f is equivalently expressed as

$$f(x) = 1, x > 0$$
$$= 0, x = 0$$
$$= -1, x < 0.$$

f is called the **signum function.**

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x], x \in \mathbb{R}$. $[x]$ is the greatest integer not greater than x. The range of the function is $\mathbb{Z}$. f is equivalently expressed as

$$f(x) = 0, 0 \le x < 1$$
$$= 1, 1 \le x < 2$$
$$= 2, 2 \le x < 3$$
$$\ldots \quad \ldots$$
$$= -1, -1 \le x < 0$$
$$= -2, -2 \le x < -1$$
$$\ldots \quad \ldots$$

f is called the **greatest integer function.**

For every $x \in \mathbb{R}$, $x \ge [x]$. The difference between x and its integral part $[x]$ is called the *fractional part* of x and is denoted by $\{x\}$.

Therefore $\{x\} = x - [x]$ for all real x. It also follows that $0 \le \{x\} < 1$ for all real x.

For example, $\{.3\} = .3$, $\{2.3\} = .3$, $\{2\} = 0$, $\{-.3\} = .7$.

Definition.

A function f defined on $I = [a, b]$ is said to be a **piecewise constant function** on I (or a **step function** on I) if there exist finite number of points $x_0, x_1, \ldots, x_n (a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b)$ such that f is a constant on each open subinterval (x_{k-1}, x_k) of $[a, b]$. That is, for each $k = 1, 2, \ldots, n$, there is a real number s_k such that $f(x) = s_k$ for all $x \in (x_{k-1}, x_k)$. $f(x_{k-1}), f(x_k)$ need not be same as s_k, $k = 1, 2, \ldots, n$.

4.2. Injective function, Surjective function.

Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *injective* (or one-one) if for two distinct elements x_1, x_2 in D the functional values $f(x_1)$ and $f(x_2)$ are distinct.

Let $D \subset \mathbb{R}, E \subset \mathbb{R}$. A function $f : D \to E$ is said to be surjective (or onto) if $f(D) = E$.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin\ x, x \in \mathbb{R}$ is not injective, because two distinct points π and 2π in the domain $\mathbb{R}$ have

the same functional value. f is not surjective, because the range of f is $\{x \in \mathbb{R} : -1 \le x \le 1\}$, a proper subset of the co-domain set.

4.3. Equal functions.

Let $D \subset \mathbb{R}$. The functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ having the same domain D are said to be *equal* if $f(x) = g(x)$ for all $x \in D$.

Examples.

1. Let $f(x) = |x|, x > 0; g(x) = x, x > 0$.
Then f and g have the same domain $\{x \in \mathbb{R} : x > 0\}$ and $f(x) = g(x)$ for all x in the domain. Therefore $f = g$.

2. Let $f(x) = \sqrt{\frac{2x}{x-1}}, x \in A \subset \mathbb{R}; g(x) = \frac{\sqrt{2x}}{\sqrt{x-1}}, x \in B \subset \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x > 1\} \cup \{x \in \mathbb{R} : x \le 0\}$, $B = \{x \in \mathbb{R} : x > 1\}$.
f and g have different domains. Therefore $f \ne g$.

4.4. Restriction function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let D_o be a non-empty subset of D. The function $g : D_o \to \mathbb{R}$ defined by $g(x) = f(x), x \in D_o$ is said to be the *restriction* of f to D_o and g is denoted by f/D_o.

Examples.

1. Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \text{ sgn } x, x \in \mathbb{R}$.

Let $D_o = \{x \in \mathbb{R} : x > 0\}$. Then the restriction function f/D_o is defined by $f/D_o\ (x) = 1, x > 0$.

Let $D_1 = \{x \in \mathbb{R} : x < 0\}$. Then the restriction function f/D_1 is defined by $f/D_1\ (x) = -1, x < 0$.

2. Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x], x \in \mathbb{R}$.

Let $D_o = \{x \in \mathbb{R} : 0 \le x < 1\}$. Then the restriction function f/D_o is defined by $f/D_o\ (x) = 0, 0 \le x < 1$.

Let $D_1 = \{x \in \mathbb{R} : 1 \le x < 2\}$. Then the restriction function f/D_1 is defined by $f/D_1\ (x) = 1, 1 \le x < 2$.

3. Let $D = \{x \in \mathbb{R} : 0 \le x \le \frac{\pi}{2}\}$ and $f : D \to \mathbb{R}$ is defined by
$f(x) = \sqrt{1 - \sin 2x}, x \in D$.

Let $D_o = \{x \in \mathbb{R} : 0 \le x \le \frac{\pi}{4}\}$. Then the restriction function f/D_o is defined by $f/D_o(x) = \cos x - \sin x, 0 \le x \le \pi/4$.

Let $D_1 = \{x \in \mathbb{R} : \frac{\pi}{4} \le x \le \frac{\pi}{2}\}$. Then the restriction function f/D_1 is defined by $f/D_1(x) = \sin x - \cos x, \frac{\pi}{4} \le x \le \frac{\pi}{2}$.

4.5. Composite function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $g : E \to \mathbb{R}$ be a function on E, where $f(D) \subset E \subset \mathbb{R}$. Then for each $x \in D$, $f(x) \in E$ and therefore $g(f(x)) \in \mathbb{R}$. We can conceive of a real function $h : D \to \mathbb{R}$ such that $h(x) = g(f(x)), x \in D$. Then h is said to be the *composite function* of f and g and the function h is expressed as gf or as $g \circ f$.

Examples.

1. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f : D \to \mathbb{R}$ is defined by $f(x) = \sqrt{x}, x \in D$, and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = e^x, x \in \mathbb{R}$.

$f(D) = \{x \in \mathbb{R} : x \geq 0\}$. $f(D)$ is a subset of the domain of g. The composite function $g \circ f : D \to \mathbb{R}$ is defined by $g \circ f(x) = e^{\sqrt{x}}, x \in D$, i.e., $g \circ f(x) = e^{\sqrt{x}}, x \geq 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1, x \in \mathbb{R}$. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $g : D \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 1\}$ and this is a subset of the domain of g.

The composite function $gf : \mathbb{R} \to \mathbb{R}$ is defined by $gf(x) = \sqrt{x^2 + 1}, x \in \mathbb{R}$.

4.6. Inverse function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be an injective function. Let $f(D) = E \subset \mathbb{R}$. Then $f : D \to E$ is injective as well as surjective.

Let $x \in D$. Then $f(x) = y \in E$. Each y in E has exactly one pre-image x in D. We can define a function $g : E \to D$ by $g(y) = x, y \in E$ where $f(x) = y$.

Therefore $gf(x) = x$ for all $x \in D$ and $fg(y) = y$ for all $y \in E$.
g is said to be the *inverse* of f and is denoted by f^{-1}.

The domain of the inverse function f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$.

Examples.

1. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f(x) = x^2, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 0\} = E$, say. Then $f : D \to E$ is injective as well as surjective.

The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \sqrt{y}, y \in E$.
Also $f^{-1}f(x) = x$ for all $x \geq 0$ and $ff^{-1}(y) = y$ for all $y \geq 0$;
i.e., $\sqrt{x^2} = x$ for all $x \geq 0$ and $(\sqrt{y})^2 = y$ for all $y \geq 0$.
This inverse function is called **the square root function**.

2. Let $D = \{x \in \mathbb{R} : x \leq 0\}$ and $f(x) = x^2, x \leq 0$. The range of f is $\{x \in \mathbb{R} : x \geq 0\} = E$, say. Then $f : D \to E$ is injective as well as surjective.

The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = -\sqrt{y}, y \in E$.
Also $f^{-1}f(x) = x$ for all $x \leq 0$ and $ff^{-1}(y) = y$ for all $y \geq 0$;
i.e., $-\sqrt{x^2} = x$ for all $x \leq 0$ and $(-\sqrt{y})^2 = y$ for all $y \geq 0$.
This inverse function is called the **negative square root** function.

Note. The function $f(x) = x^2, x \in \mathbb{R}$ admits of two inverse functions, the *principal* inverse function is the function described in Example 1.

3. The real sine function defined on $\mathbb{R}$ is not injective on $\mathbb{R}$. The range of the function is $E = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

Let us consider the subset $D = \{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$. Then $f : D \to E$ defined by $f(x) = \sin x, x \in D$ is injective as well as surjective.
The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \sin^{-1} y, y \in E$.
Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$;
i.e., $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $\sin(\sin^{-1} y) = y$ for $-1 \leq y \leq 1$.

This inverse function is called the **principal inverse sine function**. The domain of the inverse function is $\{y \in \mathbb{R} : -1 \leq y \leq 1\}$ and the range is $\{x \in \mathbb{R} : \frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$

Therefore $-\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$ for $-1 \leq y \leq 1$.

Note. If instead of D, we choose $D_1 = \{x \in \mathbb{R} : 3\pi/2 \leq x \leq 5\pi/2\}$ as the domain then the function $f(x) = \sin x, x \in D_1$, is injective as well as surjective and therefore it admits of an inverse function $f^{-1} : E \to D_1$ satisfying the conditions

$f^{-1}f(x) = x$ for all $x \in D_1$ and $ff^{-1}(y) = y$ for all $y \in E$.

But this inverse function differs from the principal inverse sine function as they have different ranges.

Equivalently, we can define *many* inverse sine functions on the same domain E with their respective ranges different. This is expressed by saying that inverse of real sine function is a *many-valued function* and this is denoted by Sin^{-1} (or Arc sin). The *principal* inverse function is denoted by $\sin^{-1}$ (or arc sin).

Thus sin (Arc sin y) $= y$, for $-1 \leq y \leq 1$ but Arc sin $(\sin x) \neq x$, in general.

4. The real cosine function $f(x) = \cos x, x \in \mathbb{R}$ is not injective. The range of the function is $E = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

Let us consider the subset $D = \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$. Then the function $f : D \to E$ defined by $f(x) = \cos x, x \in D$ is injective as well as

4.8. Monotone functions.

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **monotone increasing** on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

$f : I \to \mathbb{R}$ is said to be **monotone decreasing** on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

A function $f : I \to \mathbb{R}$ is said to be *monotone* on I if f is either monotone increasing or monotone decreasing on I.

A function $f : I \to \mathbb{R}$ is said to be *strictly increasing* on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

$f : I \to \mathbb{R}$ is said to be *strictly decreasing* on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

A function $f : I \to \mathbb{R}$ is said to be *strictly monotone* on I if f is either strictly increasing or strictly decreasing on I.

Let $I = [a, b]$ be a closed and bounded interval.

A function $f : I \to \mathbb{R}$ is said to be monotone increasing on I if $x_1, x_2 \in I$ and $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) \leq f(x_2)$.

Similar definitons for a monotone decreasing function.

Let I be an interval and $f : I \to \mathbb{R}, g : I \to \mathbb{R}$ are both monotone increasing (decreasing) on I. Then

(i) $f + g$ is monotone increasing (decreasing) on I;
(ii) if $k \in \mathbb{R}$ and $k > 0, kf$ is monotone increasing (decreasing) on I;
(iii) if $k \in \mathbb{R}$ and $k < 0, kf$ is monotone decreasing (increasing) on I.

Examples.

1. Let $f(x) = 1 - x, x \in \mathbb{R}$.
$x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.
Therefore f is strictly decreasing on $\mathbb{R}$.

2. Let $f(x) = x^2, x \in \mathbb{R}$.
$x_1, x_2 \in \mathbb{R}$ and $0 \leq x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
$x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \leq 0 \Rightarrow f(x_1) > f(x_2)$.
Therefore f is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$.

3. Let $f(x) = \text{sgn}\, x, x \in [-1, 1]$.
$x_1 < 0, x_2 < 0$ and $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.
$x_1 < 0, x_2 > 0$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
$x_1 > 0, x_2 > 0$ and $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.
Therefore f is monotone increasing on $[-1, 1]$.

4.9. Even function, odd function.

For $a \in \mathbb{R}^*$, let D be the symmetric interval $(-a, a)$.

A function $f : D \to \mathbb{R}$ is said to be an *even* function if $f(-x) = f(x)$ for all $x \in D$.

A function $f : D \to \mathbb{R}$ is said to be an *odd* function if $f(-x) = -f(x)$ for all $x \in D$.

For example,the functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2, f(x) = \cos x$ are even functions on $\mathbb{R}$ and defined by $f(x) = x, f(x) = \text{sgn}\, x, f(x) = \sin x$ are odd functions on $\mathbb{R}$.

If f be an odd function on $(-a, a)$ then $f(0) = 0$.

Let f be an odd function on $(-a, a)$ for some $a \in \mathbb{R}^*$. If $(x, f(x))$ be a point on the graph of f, then $(-x, -f(x))$ is also a point on the graph. It follows that the graph of f is symmetrical about the origin.

Let f be an even function on $(-a, a)$ for some $a \in \mathbb{R}^*$. If (x, y) be a point on the graph of f, then $(-x, y)$ is also a point on the graph. It follows that the graph of f is symmetrical about the y axis.

4.10. Power functions.

A. Positive Integral powers.

Case 1. Let n be an even positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $[0, \infty)$.

f is not injective on $\mathbb{R}$ since $f(c) = f(-c)$ for all $c \in \mathbb{R}$.

Let $x_1, x_2 \in [0, \infty)$ and $0 \leq x_1 < x_2$. Then $f(x_1) < f(x_2)$. f is a strictly increasing function on $[0, \infty)$.

Let $x_1, x_2 \in (-\infty, 0]$ and $x_1 < x_2 \leq 0$. Then $f(x_1) > f(x_2)$. f is a strictly decreasing function on $(-\infty, 0]$.

If we restrict the domain of f to $[0, \infty)$, then the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^n, x \in [0, \infty)$ is a strictly increasing function on $[0, \infty)$ and therefore f is injective on $[0, \infty)$.

For each $y \in (0, \infty)$ there exists a unique $x \in (0, \infty)$ such that $x^n = y$ [2.4.23, worked Ex.9]. This together with $f(0) = 0$ shows that f is surjective.

Therefore f is a bijective function and the inverse function f^{-1} is defined by $f^{-1}(x) = x^{\frac{1}{n}}, x \in [0, \infty)$.

This inverse function is called the *nth rootfunction* (n even positive integer) and the domain of this function is $[0, \infty)$.

Case 2. Let n be an odd positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $\mathbb{R}$.

5. SEQUENCE

5.1. Real Sequence.

A mapping $f : \mathbb{N} \to \mathbb{R}$ is said to be a *sequence in* $\mathbb{R}$, or a *real sequence.*

The f-images $f(1), f(2), f(3), \ldots \ldots$ are real numbers. The image of the nth element, $f(n)$, is said to be the nth *element* of the sequence f.

We shall be mainly concerned with real sequences and we shall use the term 'sequence' to mean a 'real sequence'.

A sequence f is generally denoted by the symbol $(f(n))$ or by the symbol $(f(n))_n$. Also the symbol $(f(1), f(2), f(3), \ldots \ldots)$ is used to denote the sequence f.

The *range* of the real sequence $(f(n))$ is a subset of $\mathbb{R}$, denoted by the symbol $\{f(n) : n \in \mathbb{N}\}$.

The symbols like $(u_n), (v_n), (x_n)$, etc. shall also be used to denote a sequence.

Examples.

1. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n, n \in \mathbb{N}$. Then $f(1) = 1, f(2) = 2, \ldots \ldots$ The sequence is denoted by (n). It is also denoted by $(n)_n$ or by $(1, 2, 3, \ldots \ldots)$.

2. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n^2, n \in \mathbb{N}$. The sequence is (n^2). It is also denoted by $(n^2)_n$ or by $(1^2, 2^2, 3^2, \ldots \ldots)$.

3. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{n}{n+1}, n \in \mathbb{N}$. The sequence is $(\frac{n}{n+1})$. It is also denoted by $(\frac{n}{n+1})_n$ or by $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \ldots)$.

4. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n, n \in \mathbb{N}$. The sequence is $((-1)^n)$. It is also denoted by $((-1)^n)_n$ or by $(-1, 1, -1, \ldots \ldots)$. The range of the sequence is $\{-1, 1\}$.

5. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \sin \frac{n\pi}{2}, n \in \mathbb{N}$. The sequence is $(1, 0, -1, 0, 1, 0, \ldots \ldots)$. The range of the sequence is $\{-1, 0, 1\}$.

6. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = 2$ for all $n \in \mathbb{N}$. The sequence is $(2, 2, 2, \ldots \ldots)$. It is called a *constant sequence* .

Sometimes it is convenient to specify $f(1)$ and describe $f(n+1)$ in terms of $f(n)$ for all $n \geq 1$.

For example, the function f defined on $\mathbb{N}$ by $f(1) = \sqrt{2}$ and $f(n+1) = \sqrt{2f(n)}$ for $n \geq 1$, is the sequence $(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \ldots)$.

5.2. Bounded Sequence.

A real sequence $(f(n))$ is said to be *bounded above* if there exists a real number G such that $f(n) \leq G$ for all $n \in \mathbb{N}$. G is said to be *an upper bound* of the sequence.

A real sequence $(f(n))$ is said to be *bounded below* if there exists a real number g such that $f(n) \geq g$ for all $n \in \mathbb{N}$. g is said to be *a lower bound* of the sequence.

A real sequence $(f(n))$ is said to be a *bounded sequence* if there exist real numbers G, g such that $g \leq f(n) \leq G$ for all $n \in \mathbb{N}$.

Therefore a real sequence is bounded if and only if it is bounded above as well as bounded below. In this case, the range of the sequence is a bounded set.

For a real sequence $(f(n))$ bounded above, the range of the sequence is a set bounded above and by the supremum property of $\mathbb{R}$, the range set has the least upper bound, which is also called the *least upper bound* of the sequence $(f(n))$ and is denoted by $\sup\{f(n)\}$.

The least upper bound of a real sequence $(f(n))$ is a real number M satisfying the following conditions :

(i) $f(n) \leq M$ for all $n \in \mathbb{N}$,

(ii) for each pre-assiged positive ϵ, there exists a *natural number* k such that $f(k) > M - \epsilon$.

By similar arguments, for a real sequence $(f(n))$ bounded below, there exists a *greatest lower bound* and it is denoted by $\inf\{f(n)\}$.

The greatest lower bound of a real sequence $(f(n))$ is a real number m satisfying the following conditions :

(i) $f(n) \geq m$ for all $n \in \mathbb{N}$,

(ii) for each pre-assigned positive ϵ, there exists a *natural number* k such that $f(k) < m + \epsilon$.

For a real sequence $(f(n))$ unbounded above, we define $\sup\{f(n)\} = \infty$.
For a real sequence $(f(n))$ unbounded below, we define $\inf\{f(n)\} = -\infty$.

Examples.

1. The sequence $(\frac{1}{n})$ is a bounded sequence. 0 is the greatest lower bound and 1 is the least upper bound of the sequence.

2. The sequence (n^2) is bounded below and unbounded above. Here $\sup\{f(n)\} = \infty, \inf\{f(n)\} = 1$.

3. The sequence $(-2n)$ is bounded above and unbounded below. Here $\sup\{f(n)\} = -2, \inf\{f(n)\} = -\infty$.

4. Let $f(n) = (-1)^n n, n \in \mathbb{N}$. The sequence $(f(n))$ is unbounded above and unbounded below. The sequence is $(-1, 2, -3, 4, \ldots \ldots)$.
Here $\sup\{f(n)\} = \infty, \inf\{f(n)\} = -\infty$.

5.3. Limit of a sequence.

Let $(f(n))$ be a real sequence. A *real number* l is said to be a *limit* of the sequence $(f(n))$ if corresponding to a pre-assigned positive ϵ, there exists a *natural number* k (depending on ϵ) such that

$$| f(n) - l | < \epsilon \text{ for all } n \geq k$$
$$\text{i.e., } l - \epsilon < f(n) < l + \epsilon \text{ for all } n \geq k.$$

To be explicit, a real number l is said to be a limit of the sequence $(f(n))$ if for a pre-assigned positive ϵ, there exists a natural number k such that *all elements* of the sequence, excepting the first $k-1$ at most, lie in the ϵ- neighbourhood of l.

Theorem 5.3.1. A sequence can have at most one limit.

Proof. If possible, let a sequence $(f(n))$ have two distinct limits l_1 and l_2 where $l_1 < l_2$.

Let $\epsilon = \frac{1}{2}(l_2 - l_1)$. Then $\epsilon > 0$ and $l_1 + \epsilon = l_2 - \epsilon$. Therefore the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ are disjoint.

Since l_1 is a limit of the sequence, for the chosen ϵ, there exists a natural number k_1 such that

$$l_1 - \epsilon < f(n) < l_1 + \epsilon \text{ for all } n \geq k_1.$$

Since l_2 is a limit of the sequence, for the same chosen ϵ, there exists a natural number k_2 such that

$$l_2 - \epsilon < f(n) < l_2 + \epsilon \text{ for all } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$.
Then $l_1 - \epsilon < f(n) < l_1 + \epsilon$ and $l_2 - \epsilon < f(n) < l_2 + \epsilon$ for all $n \geq k$.

This cannot happen since the neighbourhoods $N(l_1, \epsilon)$ and $N(l_2, \epsilon)$ are disjoint. Therefore our assumption that $l_1 \neq l_2$ is wrong.

Hence $l_1 = l_2$ and this proves the theorem.

5.4. Convergent sequence.

A real sequence $(f(n))$ is said to be a *convergent sequence*, if it has a limit $l \in \mathbb{R}$. In this case the sequence is said to converge to l.

We write $\lim_{n\to\infty} f(n) = l$, or $\lim f(n) = l$.

A sequence is said to be a *divergent sequence*, if it is not convergent.

Examples.

1. The sequence $(\frac{1}{n})$ converges to 0.

Let us choose a positive ϵ.

By Archimedean property of the set $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \epsilon$. This implies $0 < \frac{1}{n} < \epsilon$ for all $n \geq k$.

It follows that $| \frac{1}{n} - 0 | < \epsilon$ for all $n \geq k$ and this proves $\lim \frac{1}{n} = 0$.

2. The sequence $(\frac{n^2+1}{n^2})$ converges to 1.

Let us choose a positive ϵ.

$| \frac{n^2+1}{n^2} - 1 | < \epsilon$ will hold if $\frac{1}{n^2} < \epsilon$, i.e., if $n > \frac{1}{\sqrt{\epsilon}}$.

Let $k = [\frac{1}{\sqrt{\epsilon}}] + 1$. [For example, if $\epsilon = .01$ then $k = 11$; if $\epsilon = .001$ then $k = 32$].

Then k is a natural number and $| \frac{n^2+1}{n^2} - 1 | < \epsilon$ for all $n \geq k$.

This proves $\lim \frac{n^2+1}{n^2} = 1$.

3. Let $f(n) = 2$ for all $n \in \mathbb{N}$. The sequence is $(2, 2, 2, \ldots \ldots)$. We prove that the sequence converges to 2.

Let us choose a positive ϵ.

$| f(n) - 2 | < \epsilon$ holds for all $n \geq 1$.

Therefore $\lim f(n) = 2$.

Note. A constant sequence is a convergent sequence.

Theorem 5.4.1. A convergent sequence is bounded.

Proof. Let $(f(n))$ be a convergent sequence and let l be its limit. Let us choose $\epsilon = 1$. For this chosen ϵ there exists a natural number k such that $l - 1 < f(n) < l + 1$ for all $n \geq k$.

Let $B = \max\{f(1), f(2), \ldots, f(k-1), l+1\}$;

$b = \min\{f(1), f(2), \ldots, f(k-1), l-1\}$.

Then $b \leq f(n) \leq B$ for all $n \in \mathbb{N}$.

This proves that the sequence $(f(n))$ is a bounded sequence.

Corollary. An unbounded sequence is not convergent.

Note. A bounded sequence may not be a convergent sequence.

For example, the sequence $((-1)^n)$ is a bounded sequence but the sequence does not converge to a limit.

5.5. Limit theorems.

Theorem 5.5.1. Let (u_n) and (v_n) be two convergent sequences that converge to u and v respectively.

Then (i) $\lim(u_n + v_n) = u + v$;

(ii) if $c \in \mathbb{R}$, $\lim(cu_n) = cu$;

(iii) $\lim(u_n v_n) = uv$;

(iv) $\lim(\frac{u_n}{v_n}) = \frac{u}{v}$, provided (v_n) is a sequence of non zero real numbers and $v \neq 0$.

Proof. (i) To show that $\lim(u_n + v_n) = u + v$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that $| (u_n + v_n) - (u + v) | < \epsilon$ for all $n \geq k$.

Using triangle inequality, we have

$| (u_n + v_n) - (u + v) | = | (u_n - u) + (v_n - v) | < | u_n - u | + | v_n - v |$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k_1 such that $| u_n - u | < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since $\lim v_n = v$, there exists a natural number k_2 such that $| v_n - v | < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| u_n - u | < \frac{\epsilon}{2}$ and $| v_n - v | < \frac{\epsilon}{2}$ for all $n \geq k$. It follows that $| (u_n + v_n) - (u + v) | < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(u_n + v_n) = u + v$.

(ii) Let us assume $c \neq 0$. When $c = 0$ the theorem is obvious.

To show that $\lim cu_n = cu$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$| cu_n - cu | < \epsilon$ for all $n \geq k$.

We have $| cu_n - cu | = | c || u_n - u |$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k such that $| u_n - u | < \frac{\epsilon}{|c|}$ for all $n \geq k$.

It follows that $| cu_n - cu | < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(cu_n) = cu$.

(iii) To show that $\lim(u_n v_n) = uv$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that $| u_n v_n - uv | < \epsilon$ for all $n \geq k$.

We have $| u_n v_n - uv | = | u_n(v_n - v) + v(u_n - u) |$

$\leq |u_n||v_n - v| + |v||u_n - u|$.

Since (u_n) is a convergent sequence, it is bounded. Therefore there exists a positive real number B_1 such that $|u_n| < B_1$ for all $n \in \mathbb{N}$.

Let $B = \max\{B_1, |v|\}$.

Then $|u_n v_n - uv| < B|v_n - v| + B|u_n - u|$.

Let $\epsilon > 0$. Since $\lim u_n = u$ and $\lim v_n = v$, there exist natural numbers k_1 and k_2 such that

$|u_n - u| < \frac{\epsilon}{2B}$ for all $n \geq k_1$ and $|v_n - v| < \frac{\epsilon}{2B}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|u_n - u| < \frac{\epsilon}{2B}$ and $|v_n - v| < \frac{\epsilon}{2B}$ for all $n \geq k$.

It follows that $|u_n v_n - uv| < B.\frac{\epsilon}{2B} + B.\frac{\epsilon}{2B}$ for all $n \geq k$

or, $|u_n v_n - uv| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(u_n v_n) = uv$.

(iv) First we prove that if $\lim v_n = v$, where (v_n) is a sequence of non-zero real numbers and $v \neq 0, \lim 1/v_n = 1/v$.

Let $\alpha = \frac{1}{2}|v|$. Then $\alpha > 0$. Since $\lim v_n = v$, there exists a natural number k_1 such that $|v_n - v| < \alpha$ for all $n \geq k_1$.

We have $||v_n| - |v|| \leq |v_n - v| < \alpha$ for all $n \geq k_1$

or, $|v| - \alpha < |v_n| < |v| + \alpha$ for all $n \geq k_1$.

Therefore $|v_n| > \frac{1}{2}|v|$ for all $n \geq k_1$.

$\left|\frac{1}{v_n} - \frac{1}{v}\right| = \frac{|v_n - v|}{|v||v_n|} < \frac{2}{|v|^2}|v_n - v|$ for all $n \geq k_1$.

Let $\epsilon > 0$. Since $\lim v_n = v$, there exists a natural number k_2 such that $|v_n - v| < \frac{|v|^2}{2}\epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $\left|\frac{1}{v_n} - \frac{1}{v}\right| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim \frac{1}{v_n} = \frac{1}{v}$.

The proof of the theorem is now completed by considering the convergence of the product of two sequences (u_n) and $(\frac{1}{v_n})$.

Therefore $\lim(\frac{u_n}{v_n}) = \lim(u_n.\frac{1}{v_n}) = u.\frac{1}{v} = \frac{u}{v}$.

Note. If $(u_n), (v_n), (w_n)$ be three convergent sequences of real numbers that converge to u, v, w respectively, then

(i) $\lim(u_n + v_n + w_n) = u + v + w$ and

(ii) $\lim(u_n v_n w_n) = uvw$.

The theorem can be generalised to the sum and the product of a *finite* number of convergent sequences.

Theorem 5.5.2. *Let (u_n) be a convergent sequence of real numbers converging to u. Then the sequence $(|u_n|)$ converges to $|u|$.*

Proof. We have $||u_n| - |u|| \leq |u_n - u|$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k such that $|u_n - u| < \epsilon$ for all $n \geq k$.

It follows that $||u_n| - |u|| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim |u_n| = |u|$.

Note 1. The converse of the theorem is not true. That is, if the sequence $(|u_n|)$ is a convergent sequence, it does not necessarily imply that the sequence (u_n) is a convergent sequence.

For example, let $u_n = (-1)^n$. Then the sequence $(|u_n|)$ converges to 1 but the sequence (u_n) is a divergent sequence.

Theorem 5.5.3. Let (u_n) be a convergent sequence of real numbers and there exists a natural number m such that $u_n > 0$ for all $n \geq m$. Then $\lim u_n \geq 0$.

Proof. Let $\lim u_n = u$ and if possible let $u < 0$.

Let us choose a positive ϵ such that $u + \epsilon < 0$.

Since $\lim u_n = u$, there exists a natural number k_1 such that

$$u - \epsilon < u_n < u + \epsilon \text{ for all } n \geq k_1.$$

Let $k = \max\{k_1, m\}$.

Then by hypothesis, $u_n > 0$ for all $n \geq k$ and we have from above $u_n < u + \epsilon < 0$ for all $n \geq k$.

This is a contradiction. Therefore $\lim u_n \geq 0$.

Note 1. The theorem also says that a convergent sequence of positive real numbers may converge to 0. For example, if $u_n = \frac{1}{n}$, then (u_n) is a convergent sequence of positive real numbers but $\lim u_n = 0$.

Note 2. If (u_n) be a convergent sequence and $u_n \geq 0$ for all $n \geq m$ (m being a natural number), then $\lim u_n \geq 0$.

Theorem 5.5.4. Let (u_n) and (v_n) be two convergent sequences and there exists a natural number m such that $u_n > v_n$ for all $n \geq m$.

Then $\lim u_n \geq \lim v_n$.

Proof. Let $\lim u_n = u, \lim v_n = v$ and $w_n = u_n - v_n$.

Then (w_n) is a convergent sequence such that $w_n > 0$ for all $n \geq m$ and $\lim w_n = u - v$.

By the previous theorem, $u - v \geq 0$.

Consequently, $\lim u_n \geq \lim v_n$.

Note. If (u_n) and (v_n) be two convergent sequences and $u_n \geq v_n$ for all $n \geq m$, then $\lim u_n \geq \lim v_n$.

Let $w_n = u_n - v_n$. Then (w_n) is a convergent sequence such that $w_n \geq 0$ for all $n \geq m$ and $\lim w_n = u - v$.
So $u - v \geq 0$ and therefore $\lim u_n \geq \lim v_n$.

Corollary. If (x_n) is a convergent sequence of points in a closed and bounded interval $[a, b]$ and $\lim x_n = c$, then $c \in [a, b]$.
$a \leq x_n \leq b$ implies $a \leq \lim x_n \leq b$, i.e., $a \leq c \leq b$.

Note. If (x_n) is a convergent sequence of points in an open bounded interval (a, b) and $\lim x_n = c$, then c may not be in (a, b).

For example, the sequence $(\frac{1}{2n})$ is a convergent sequence in the open interval $(0, 1)$, but the limit of the sequence does not belong to $(0, 1)$.

Theorem 5.5.5. Sandwich theorem. Squeeze theorem.

Let $(u_n), (v_n), (w_n)$ be three sequences of real numbers and there is a natural number m such that $u_n < v_n < w_n$ for all $n \geq m$.
If $\lim u_n = \lim w_n = l$, then the sequence (v_n) is convergent and $\lim v_n = l$.

Proof. Let $\epsilon > 0$. It follows from the convergence of the sequences (u_n) and (w_n) that there exist natural numbers k_1 and k_2 such that
$| u_n - l | < \epsilon$ for all $n \geq k_1$ and $| w_n - l | < \epsilon$ for all $n \geq k_2$.

Let $k_3 = \max\{k_1, k_2\}$.
Then $l - \epsilon < u_n < l + \epsilon$ and $l - \epsilon < w_n < l + \epsilon$ for all $n \geq k_3$.

Let $k = \max\{k_3, m\}$.
Then $l - \epsilon < u_n < v_n < w_n < l + \epsilon$ for all $n \geq k$.
Consequently, $| v_n - l | < \epsilon$ for all $n \geq k$.

This shows that the sequence (v_n) is convergent and $\lim v_n = l$.
This completes the proof.

Note. If $u_n \leq v_n \leq w_n$ for all $n \geq m$ and $\lim u_n = \lim w_n = l$, then $\lim v_n = l$.

Worked Examples.

1. Prove that $\lim\limits_{n \to \infty} \frac{3n^2+2n+1}{n^2+1} = 3$.

$\lim\limits_{n \to \infty} \frac{3n^2+2n+1}{n^2+1} = \lim\limits_{n \to \infty} \frac{u_n}{v_n}$, where $u_n = 3 + \frac{2}{n} + \frac{1}{n^2}$ and $v_n = 1 + \frac{1}{n^2}$.
But $\lim u_n = 3$ and $\lim v_n = 1$.
Therefore $\lim\limits_{n \to \infty} \frac{3n^2+2n+1}{n^2+1} = \lim \frac{u_n}{v_n} = 3$.

2. Prove that $\lim\limits_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

$$\begin{aligned}
\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) &= \lim_{n \to \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&= \lim_{n \to \infty} u_n v_n, \text{ where } u_n = \tfrac{1}{\sqrt{n}}, v_n = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\
&= 0, \text{ since } \lim u_n = 0 \text{ and } \lim v_n = \tfrac{1}{2}.
\end{aligned}$$

3. Prove that
$\lim\limits_{n \to \infty} (\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}) = 1$.

Let $u_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$.

$$\begin{aligned}
\text{We have } \frac{1}{\sqrt{n^2+2}} &< \frac{1}{\sqrt{n^2+1}} \\
\frac{1}{\sqrt{n^2+3}} &< \frac{1}{\sqrt{n^2+1}} \\
\cdots & \quad \cdots \\
\frac{1}{\sqrt{n^2+n}} &< \frac{1}{\sqrt{n^2+1}}.
\end{aligned}$$

Therefore $u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

$$\begin{aligned}
\text{Again, } \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} &> \frac{2}{\sqrt{n^2+2}} \\
\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} &> \frac{3}{\sqrt{n^2+3}} \\
\cdots \quad \cdots \quad \cdots & \quad \cdots
\end{aligned}$$

Therefore $u_n > \frac{n}{\sqrt{n^2+n}}$ for all $n \geq 2$.

Thus $\frac{n}{\sqrt{n^2+n}} < u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

But $\lim\limits_{n \to \infty} \frac{n}{\sqrt{n^2+n}} = 1$ and $\lim\limits_{n \to \infty} \frac{n}{\sqrt{n^2+1}} = 1$.

By Sandwich theorem, $\lim u_n = 1$.

5.6. Null sequence.

Definition. A sequence (u_n) is said to be a *null sequence* if $\lim u_n = 0$.

Theorem 5.6.1. If the sequence (u_n) be a null sequence, then the sequence $(| u_n |)$ is a null sequence and conversely.

Proof. Let $\epsilon > 0$. Since $\lim u_n = 0$, there exists a natural number k such that $| u_n | < \epsilon$ for all $n \geq k$.
As $| | u_n | - 0 | = | u_n |$, it follows that $| | u_n | - 0 | < \epsilon$ for all $n \geq k$.
This proves $\lim | u_n | = 0$.

Conversely, let $\lim | u_n | = 0$.
Let $\epsilon > 0$. There exists a natural number k such that $| | u_n | - 0 | < \epsilon$ for all $n \geq k$. That is, $| u_n | < \epsilon$ for all $n \geq k$.
This proves $\lim u_n = 0$.

5.7. Divergent sequence.

A real sequence $(f(n))$ is said to *diverge* to ∞ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that

$f(n) > G$ for all $n \geq k$.

In this case we write $\lim f(n) = \infty$ and also say that the sequence $(f(n))$ tends to ∞.

A real sequence $(f(n))$ is said to diverge to $-\infty$ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that

$f(n) < -G$ for all $n \geq k$.

In this case we write $\lim f(n) = -\infty$ and also say that the sequence $(f(n))$ tends to $-\infty$.

Definition. A real sequence $(f(n))$ is said to be a *properly divergent sequence* if it either diverges to ∞, or diverges to $-\infty$.

Theorem 5.7.1. A sequence diverging to ∞ is unbounded above but bounded below.

Proof. Let a sequence $(f(n))$ diverge to ∞. Then for each pre-assigned positive number G there exists a natural number k such that $f(k) > G$.

Therefore there does not exist a real number B such that $f(n) \leq B$ holds for all $n \in \mathbb{N}$. In other words, $(f(n))$ is unbounded above.

Let $G > 0$. Then there exists a natural number k such that

$f(n) > G$ for all $n \geq k$.

Let $b = \min\{f(1), f(2), \ldots, f(k-1), G\}$. Then $f(n) \geq b$ for all $n \in \mathbb{N}$.

This proves that the sequence $(f(n))$ is bounded below.

Note. A sequence unbounded above but bounded below may not diverge to ∞.

For example, let us consider the sequence $(f(n))$ where $f(n) = n^{(-1)^n}$. The sequence is $(1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots \ldots)$.

The sequence is unbounded above and bounded below, 0 being a lower bound. The sequence does not diverge to ∞, because for a pre-assigned positive number G there does not exist a natural number k such that $f(n) > G$ holds for all $n \geq k$.

Theorem 5.7.2. A sequence diverging to $-\infty$ is unbounded below but bounded above.

Proof left to the reader.

Note. A sequence unbounded below but bounded above may not diverge to $-\infty$.

Definitions. A bounded sequence that is not convergent is said to be an *oscillatory sequence of finite oscillation.*

An unbounded sequence that is not properly divergent is said to be an *oscillatory sequence of infinite oscillation.*

An oscillatory sequence is therefore neither convergent nor properly divergent. It is called an *improperly divergent sequence.*

Examples.

1. The sequence (2^n) diverges to ∞.

2. The sequence $(-n^2)$ diverges to $-\infty$.

3. The sequence $((-1)^n)$ is a bounded sequence, but not convergent. It is an oscillatory sequence of finite oscillation.

4. The sequence $((-1)^n n)$ is an unbounded sequence, and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

5.8. Some important limits.

1. $\lim r^n = 0$ if $|r| < 1$.

Case 1. $r = 0$. In this case the sequence is $(0, 0, 0, \ldots \ldots)$.
The sequence converges to 0. That is, $\lim r^n = 0$ when $r = 0$.

Case 2. $r \neq 0$ and $|r| < 1$.

$\frac{1}{|r|} > 1$, since $|r| < 1$. Let $\frac{1}{|r|} = a + 1$, where $a > 0$.

$|r^n - 0| = |r^n| = |r|^n = \frac{1}{(a+1)^n}$.

We have $(1+a)^n > na$ for all $n \in \mathbb{N}$. So $|r^n - 0| < \frac{1}{na}$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Then $|r^n - 0| < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.

Let $k = [\frac{1}{a\epsilon}] + 1$. Then k is a natural number and $|r^n - 0| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim r^n = 0$.

Combining the cases, $\lim r^n = 0$ if $|r| < 1$.

2. $\lim a^{1/n} = 1$ if $a > 0$.

Case 1. $a = 1$. In this case the sequence converges to 1.

Case 2. $a > 1$. Then $a^{1/n} > 1$. Let $a^{1/n} = 1 + x_n$ where $x_n > 0$.

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Let $G > 0$. Then there exists a natural number k such that

$f(n) > G$ for all $n \geq k$.

Let $b = \min\{f(1), f(2), \ldots, f(k-1), G\}$. Then $f(n) \geq b$ for all $n \in \mathbb{N}$.

This proves that the sequence $(f(n))$ is bounded below.

Note. A sequence unbounded above but bounded below may not diverge to ∞.

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We have $(1 + a)^n > na$ for all $n \in \mathbb{N}$. So $|r^n - 0| < \frac{1}{na}$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Then $|r^n - 0| < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.

Let $k = [\frac{1}{a\epsilon}] + 1$. Then k is a natural number and $|r^n - 0| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim r^n = 0$.

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Case 1. $a = 1$. In this case the sequence converges to 1.

Case 2. $a > 1$. Then $a^{1/n} > 1$. Let $a^{1/n} = 1 + x_n$ where $x_n > 0$.

5.11. Subsequence.

Let (u_n) be a real sequence and (r_n) be a strictly increasing sequence of natural numbers, i.e., $r_1 < r_2 < r_3 < \cdots < r_n < \cdots$ Then the sequence $(u_{r_n})_n$ is said to be a *subsequence* of the sequence (u_n). The elements of the subsequence $(u_{r_n})_n$ are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots \ldots$

Let $r : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers such that $r_1 < r_2 < \cdots < r_n < \cdots \cdots$ and $u : \mathbb{N} \to \mathbb{R}$ be a real sequence. Then the composite mapping $uor : \mathbb{N} \to \mathbb{R}$ is said to be a *subsequence* of the real sequence u. The elements of the subsequence uor are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots \ldots$

Examples.

1. Let $u_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $(u_{r_n})_n = \{u_2, u_4, u_6, \ldots \ldots\}$
   $= \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots \ldots\}$ is a subsequence of $(\frac{1}{n})$.

2. Let $u_n = \frac{1}{n}$ and $r_n = 2n - 1$ for all $n \in \mathbb{N}$.
   Then $(u_{r_n})_n = (u_1, u_3, u_5, \ldots \ldots\}$
   $= \{1, \frac{1}{3}, \frac{1}{5}, \ldots \ldots\}$ is a subsequence of $(\frac{1}{n})$.

3. Let $u_n = (-1)^n$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $(u_{r_n})_n = (u_2, u_4, u_6, \ldots \ldots)$
   $= (1, 1, 1, \ldots \ldots)$ is a subsequence of $((-1)^n)$.

4. Let $u_n = 1 + 1/n$ and $r_n = n^2$ for all $n \in \mathbb{N}$.
   Then $(u_{r_n})_n = (1{+}1, 1{+}\frac{1}{2^2}, 1{+}\frac{1}{3^2}, \cdots \cdots\}$ is a subsequence of $(1{+}\frac{1}{n})$.

Theorem 5.11.1. If a sequence (u_n) converges to l, then every subsequence of (u_n) also converges to l.

Proof. Let (r_n) be a strictly increasing sequence of natural numbers. Then $(u_{r_n})_n$ is subsequence of the sequence (u_n).

Let $\epsilon > 0$. Since $\lim u_n = l$, there exists a natural number k such that $l - \epsilon < u_n < l + \epsilon$ for all $n \geq k$.

Since (r_n) is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$.

Therefore $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k_0$. This shows that $\lim\limits_{n \to \infty} u_{r_n} = l$.

Note. If there exist two different subsequences $(u_{r_n})_n$ and $(u_{k_n})_n$ of a sequence (u_n) such that $(u_{r_n})_n$ and $(u_{k_n})_n$ converge to two different limits, then the sequence (u_n) is not convergent.

If a sequence (u_n) has a divergent subsequence, then (u_n) is divergent.

Worked Examples.

1. Prove that $\lim(1 + \frac{1}{2n})^n = \sqrt{e}$.

Let $u_n = (1 + \frac{1}{n})^n, v_n = (1 + \frac{1}{2n})^{2n}$ and $w_n = (1 + \frac{1}{2n})^n$ for all $n \in \mathbb{N}$. The sequence (u_n) is a convergent sequence and $\lim u_n = e$.

Since $v_n = u_{2n}$ for all $n \in \mathbb{N}$, the sequence (v_n) is a subsequence of the sequence (u_n) and therefore $\lim v_n = e$.

$w_n = \sqrt{v_n}$ for all $n \in \mathbb{N}$. Therefore $\lim w_n = \lim \sqrt{v_n} = \sqrt{e}$.

2. Prove that the sequence $((-1)^n)$ is divergent.

Let $u_n = (-1)^n, v_n = u_{2n}, w_n = u_{2n-1}$.
Then (v_n) is the subsequence $(1, 1, 1, \ldots \ldots)$ and $\lim v_n = 1$;
(w_n) is the subsequence $(-1, -1, -1, \ldots \ldots)$ and $\lim w_n = -1$.

Since two different subsequences of the sequence (u_n) converge to two different limits, the sequence (u_n) is divergent.

Theorem 5.11.2. If the subsequences $(u_{2n})_n$ and $(u_{2n-1})_n$ of a sequence (u_n) converge to the same limit l, then the sequence (u_n) is convergent and $\lim u_n = l$.

Proof. Let us choose $\epsilon > 0$. Since $\lim u_{2n} = l$, there exists a natural number k_1 such that $| u_{2n} - l | < \epsilon$ for all $n \geq k_1$.

Since $\lim u_{2n-1} = l$, there exists a natural number k_2 such that $| u_{2n-1} - l | < \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then k is a natural number and for all $n \geq k$, $l - \epsilon < u_{2n} < l + \epsilon$ and $l - \epsilon < u_{2n-1} < l + \epsilon$.

That is, $l - \epsilon < u_n < l + \epsilon$ for all $n \geq 2k - 1$.

As $2k - 1$ is a natural number, it follows that $\lim u_n = l$.

Note 1. If two subsequences of a sequence (u_n) converge to the same limit l, the sequence (u_n) may not be convergent.

For example, let $u_n = \sin \frac{n\pi}{4}$.
The subsequence $(u_{8n-7})_n$ is $(\sin \frac{\pi}{4}, \sin \frac{9\pi}{4}, \sin \frac{17\pi}{4}, \cdots)$.
The subsequence $(u_{8n-5})_n$ is $(\sin \frac{3\pi}{4}, \sin \frac{11\pi}{4}, \sin \frac{19\pi}{4}, \cdots)$.
Both these subsequences converge to $\frac{1}{\sqrt{2}}$. But the sequence (u_n) is not convergent.

2. If three subsequences $(u_{3n})_n, (u_{3n-1})_n, (u_{3n-2})_n$ of a sequence (u_n) converge to the same limit l, then (u_n) is convergent and $\lim u_n = l$.

If k subsequences $(u_{kn})_n, (u_{kn-1})_n, (u_{kn-2})_n, \cdots (u_{kn-k+1})_n$ of a sequence (u_n) converge to the same limit l, then (u_n) converges to l.

The subsequence $(u_1, u_5, u_9, u_{13}, \ldots)$ is a monotone subsequence of the sequence (u_n).

2. Let $u_n = n^{(-1)^n}$. The sequence is $(1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \cdots)$.

Here the sequence (u_n) has no peak. Since u_{s_1} is not a peak, there is a u_1 is not a peak. Let $s_1 = 1$. Here $s_2 = 2$. natural number $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$.

Since u_{s_2} is not a peak, there is a natural number $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$. Here $s_3 = 4$.

By similar arguments, $s_4 = 6, s_5 = 8, \ldots$

Thus $(u_1, u_2, u_4, u_6, u_8, \ldots)$ is a monotone increasing subsequence of the sequence (u_n).

5.12. Subsequential limit.

Let (u_n) be a real sequence. A real number l is said to be a *subsequential limit* of the sequence (u_n) if there exists a subsequence of (u_n) that converges to l.

Theorem 5.12.1. A real number l is a subsequential limit of a sequence (u_n) if and only if every neighbourhood of l contains infinitely many elements of the sequence (u_n).

Proof. Let l be a subsequential limit of the sequence (u_n). Then there exists a subsequence $(u_{r_n})_n$ such that $\lim\limits_{n \to \infty} u_{r_n} = l$.

Let us choose a positive ϵ. Then there exists a natural number k such that $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k$.

Therefore $l - \epsilon < u_n < l + \epsilon$ for infinitely many values of n.

Since ϵ is arbitrary, every neighbourhood of l contains infinite number of elements of the sequence (u_n).

Conversely, let the sequence (u_n) be such that for each pre-assigned positive ϵ the ϵ-neighbourhood of l contains infinitely many elements of the sequence.

Let $\epsilon = 1$. Then $l - 1 < u_n < l + 1$ for infinitely many values of n. Therefore the set $S_1 = \{n : l - 1 < u_n < l + 1\}$ is an infinite subset of the set $\mathbb{N}$. By the well ordering property of the set $\mathbb{N}$, S_1 has a least element, say r_1. Therefore $l - 1 < u_{r_1} < l + 1$.

Let $\epsilon = \frac{1}{2}$. Then $l - \frac{1}{2} < u_n < l + \frac{1}{2}$ for infinitely many values of n. Therefore the set $S_2 = \{n : l - \frac{1}{2} < u_n < l + \frac{1}{2}\}$ is an infinite subset of $\mathbb{N}$ and hence there exists a natural number $r_2 (> r_1)$ in S_2 such that $l - \frac{1}{2} < u_{r_2} < l + \frac{1}{2}$.

Continuing thus, we obtain a strictly increasing sequence of natural numbers $(r_1, r_2, r_3, \ldots \ldots)$ such that $l - \frac{1}{n} < u_{r_n} < l + \frac{1}{n}$ for all $n \in \mathbb{N}$.

By Sandwich theorem, $\lim u_{r_n} = l$.

In other words the subsequence $(u_{r_n})_n$ converges to l.

That is, l is a subsequential limit of the sequence (u_n).

Note. The limit of a sequence, if it exists, is also a subsequential limit of the sequence.

Theorem 5.12.2. Bolzano-Weierstrass theorem.

Every bounded sequence of real numbers has a convergent subsequence.

Proof. Let (u_n) be a bounded sequence. Then there is a closed and bounded interval, say $I = [a, b]$, such that $u_n \in I$ for every $n \in \mathbb{N}$.

Let $c = \frac{a+b}{2}$ and $I' = [a, c], I'' = [c, b]$. Then at least one of the intervals I' and I'' contains infinitely many elements of (u_n).

Let $I_1 = [a_1, b_1]$ be such an interval. Then $I_1 \subset I$ and $\mid I_1 \mid$ = the length of the interval $= \frac{1}{2}(b - a)$.

Let $c_1 = \frac{a_1 + b_1}{2}$ and $I_1' = [a_1, c_1], I_1'' = [c_1, b_1]$. Then at least one of the intervals I_1' and I_1'' contains infinitely many elements of (u_n). Let $I_2 = [a_2, b_2]$ be such an interval.

Then $I_2 \subset I_1$ and $\mid I_2 \mid = \frac{1}{2} \mid I_1 \mid$.

Continuing thus, we obtain a sequence of closed and bounded intervals (I_n) such that

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$;

(ii) $\mid I_n \mid = \frac{1}{2^n}(b - a)$ and therefore $\lim\limits_{n \to \infty} \mid I_n \mid = 0$; and

(iii) each I_n contains infinitely many elements of (u_n).

By Cantor's theorem on nested intervals, there exists a unique point α such that $\alpha \in \bigcap\limits_{n=1}^{\infty} I_n$.

We prove that α is a subsequential limit of the sequence (u_n).

Let us choose $\epsilon > 0$. There exists a natural number k such that $0 < \frac{b-a}{2^k} < \epsilon$. That is, $\mid I_k \mid < \epsilon$.

Since $\alpha \in I_k$ and $\mid I_k \mid < \epsilon$, I_k is entirely contained in the neighbourhood $(\alpha - \epsilon, \alpha + \epsilon)$ and consequently, the ϵ-neighbourhood of α contains infinitely many elements of (u_n).

Since ϵ is arbitrary, each neighbourhood of α contains infinitely many elements of (u_n). Therefore α is a subsequential limit of (u_n).

So in this case $\underline{\lim}\, u_n + \underline{\lim}\, v_n < \underline{\lim}\,(u_n + v_n)$ and $\overline{\lim}\, u_n + \overline{\lim}\, v_n > \overline{\lim}\,(u_n + v_n)$.

Theorem 5.13.3. Let (u_n) and (v_n) be bounded sequences and $u_n > 0, v_n > 0$ for all $n \in \mathbb{N}$. Then

(i) $\overline{\lim}\, u_n . \overline{\lim}\, v_n \geq \overline{\lim}\,(u_n.v_n)$

(ii) $\underline{\lim}\, u_n . \underline{\lim}\, v_n \leq \underline{\lim}\,(u_n.v_n)$.

Proof. (i) The sequence $(u_n.v_n)$ is a bounded sequence. Let $\overline{\lim}\, u_n = l_1$, $\overline{\lim}\, v_n = l_2$, $\overline{\lim}\,(u_n.v_n) = p$.

$u_n v_n - l_1 l_2 = u_n(v_n - l_2) + l_2(u_n - l_1)$.

Since (u_n) is a bounded sequence and $u_n > 0$ for all $n \in \mathbb{N}$, there exists a positive real number B_1 such that $u_n < B_1$ for all $n \in \mathbb{N}$.

Since $\overline{\lim}\, v_n = l_2$ and $v_n > 0$ for all $n \in \mathbb{N}$, $l_2 > 0$.

Let $B = \max\{B_1, l_2\}$. Then $u_n v_n - l_1 l_2 < B(v_n - l_2) + B(u_n - l_1)$.

Let $\epsilon > 0$.

Since $\overline{\lim}\, u_n = l_1$, there exists a natural number k_1 such that $u_n < l_1 + \epsilon$ for all $n \geq k_1$.

Since $\overline{\lim}\, v_n = l_2$, there exists a natural number k_2 such that $v_n < l_2 + \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $u_n < l_1 + \epsilon$ and $v_n < l_2 + \epsilon$ for all $n \geq k$.

Therefore $u_n v_n - l_1 l_2 < B.\frac{\epsilon}{2B} + B.\frac{\epsilon}{2B}$ for all $n \geq k$.

That is, $u_n v_n < l_1 l_2 + \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, every subsequential limit of the sequence $(u_n v_n)$ is less than or equal to $l_1 l_2$.

Hence $\overline{\lim}\,(u_n.v_n) \leq l_1 l_2$, i.e., $\overline{\lim}\,(u_n.v_n) \leq \overline{\lim}\, u_n.\overline{\lim}\, v_n$.

(ii) proof left to the reader.

5.14. Cauchy criterion.

We discussed several methods of establishing convergence of a real sequence. In most of the methods, a prior knowledge of the limit is necessary. If however a sequence is monotone, the convergence can be established without any pre-conceived limit.

Cauchy's method of establishing convergence of a sequence does not require any knowledge of its limit, nor does it require the sequence to be monotone.

The method is very powerful as it is concerned only with the elements of the sequence.

Theorem 5.14.1. Cauchy's general principle of convergence.
A necessary and sufficient condition for the convergence of a sequence (u_n) is that for a pre-assigned positive ϵ there exists a natural number m such that $|\, u_{n+p} - u_n \,| < \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \ldots \ldots$

Proof. Let (u_n) be convergent and $\lim u_n = l$. Then for a pre-assigned positive ϵ there exists a natural number m such that

$$|\, u_n - l \,| < \tfrac{\epsilon}{2} \text{ for all } n \geq m.$$

Therefore $|\, u_{n+p} - l \,| < \frac{\epsilon}{2}$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

$$\begin{aligned} \text{Now } |\, u_{n+p} - u_n \,| &\leq |\, u_{n+p} - l \,| + |\, u_n - l \,| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } n \geq m \text{ and } p = 1, 2, 3, \ldots \end{aligned}$$

That is, $|\, u_{n+p} - u_n \,| < \epsilon$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

This proves that the condition is necessary.

We now prove that the sequence (u_n) is convergent under the stated condition. First we prove that the sequence (u_n) is bounded.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$|\, u_{n+p} - u_n \,| < 1 \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Therefore $|\, u_{k+p} - u_k \,| < 1$ for $p = 1, 2, 3, \ldots \ldots$

or, $u_k - 1 < u_{k+p} < u_k + 1$ for $p = 1, 2, 3, \ldots \ldots$

Let $B = \max\{u_1, u_2, \ldots, u_k, u_k + 1\}$, $b = \min\{u_1, u_2, \ldots, u_k, u_k - 1\}$. Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$, showing that (u_n) is a bounded sequence.

By Bolzano-Weierstrass theorem, the sequence (u_n) has a convergent subsequence. Let l be the limit of that subsequence. Then l is a subsequential limit of (u_n).

Let $\epsilon > 0$. Then by the given condition, there exists a natural number m such that $|\, u_{n+p} - u_n \,| < \frac{\epsilon}{3}$ for all $n \geq m$ and $p = 1, 2, 3, \ldots$

Taking $m = n$, it follows that

$$|\, u_{m+p} - u_m \,| < \tfrac{\epsilon}{3} \text{ for } p = 1, 2, 3, \ldots \ldots \ldots \quad \text{(i)}$$

Since l is a subsequential limit of (u_n), each ϵ-neighbourhood of l contains infinite number of elements of (u_n). Therefore there exists a natural number $q > m$ such that $|\, u_q - l \,| < \frac{\epsilon}{3}$.

As $q > m$, it follows from (i) that $|\, u_q - u_m \,| < \frac{\epsilon}{3}$.

$$\begin{aligned} |\, u_{m+p} - l \,| &\leq |\, u_{m+p} - u_m \,| + |\, u_m - u_q \,| + |\, u_q - l \,| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \text{ for } p = 1, 2, 3, \ldots \ldots \end{aligned}$$

Therefore $|\, u_n - l \,| < \epsilon$ for all $n \geq m + 1$.

Since ϵ is arbitrary, the sequence (u_n) converges to l.

In other words, the sequence (u_n) is a convergent sequence. This completes the proof.

Note. The condition stated in the theorem is called the "*Cauchy condition*" for convergence of a sequence.

Therefore a sequence (u_n) is convergent if and only if the *Cauchy condition* is satisfied.

Worked Examples.

1. Use Cauchy's general principle of convergence to prove that the sequence $(\frac{n}{n+1})$ is convergent.

Let $u_n = \frac{n}{n+1}$. Let p be a natural number.

Then $u_{n+p} = \frac{n+p}{n+p+1}$.

$$\begin{aligned} | u_{n+p} - u_n | &= \left| \frac{n+p}{n+p+1} - \frac{n}{n+1} \right| \\ &= \frac{p}{(n+p+1)(n+1)} \\ &< \frac{1}{n+1} < \frac{1}{n} \text{ for all } p, \text{ since } \frac{p}{n+p+1} < 1 \text{ for all } p. \end{aligned}$$

Let $\epsilon > 0$. Then $\frac{1}{n} < \epsilon$ holds for $n > \frac{1}{\epsilon}$.

Let $m = [\frac{1}{\epsilon}] + 1$. Then m is a natural number and $| u_{n+p} - u_n | < \epsilon$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

This proves that the sequence u_n is convergent.

2. Use Cauchy's general principle of convergence to prove that the sequence (u_n) where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is not convergent.

Let p be a natural number.

$| u_{n+p} - u_n | = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}$.

Let us choose $n = m$ and $p = m$.

$$\begin{aligned} \text{Then } | u_{2m} - u_m | &= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \\ &> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} \\ &= \frac{1}{2}. \end{aligned}$$

If we choose $\epsilon = \frac{1}{2}$ then no natural number k can be found such that $| u_{n+p} - u_n | < \epsilon$ will hold for all $n \geq k$ and for every natural number p.

This shows that Cauchy condition is not satisfied by the sequence and the sequence (u_n) is not convergent.

Cauchy sequence.

Definition. A sequence (u_n) is said to be a *Cauchy sequence* if for a pre-assigned positive ϵ there exists a natural number k such that

$$| u_m - u_n | < \epsilon \text{ for all } m, n \geq k.$$

Replacing m by $n + p$ where $p = 1, 2, 3, \ldots \ldots$ the above condition can be equivalently stated as

$$| u_{n+p} - u_n | < \epsilon \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots \ldots$$

Theorem 5.14.2. A convergent sequence is a Cauchy sequence.

Proof. Let (u_n) be a convergent sequence and let $\lim u_n = l$.

For a pre-assigned positive ϵ there exists a natural number k such that $| u_n - l | < \frac{\epsilon}{2}$ for all $n \geq k$.

If m, n be natural numbers $\geq k$, then

$$| u_m - l | < \tfrac{\epsilon}{2} \text{ and } | u_n - l | < \tfrac{\epsilon}{2}.$$

$$\begin{aligned} \text{Now } | u_m - u_n | &\leq | u_m - l | + | l - u_n | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } m, n \geq k. \end{aligned}$$

That is, $| u_m - u_n | < \epsilon$ for all $m, n \geq k$.

This proves that the sequence (u_n) is a Cauchy sequence.

Theorem 5.14.3. A Cauchy sequence of real numbers is convergent.

Proof. Let (u_n) be a Cauchy sequence. First we prove that the sequence (u_n) is bounded.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$| u_m - u_n | < 1 \text{ for all } m, n \geq k.$$

Therefore $| u_k - u_n | < 1$ for all $n \geq k$.

or, $u_k - 1 < u_n < u_k + 1$ for all $n \geq k$.

Let $B = \max\{u_1, u_2, \ldots, u_{k-1}, u_k + 1\}$,

$b = \min\{u_1, u_2, \ldots, u_{k-1}, u_k - 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$ and this proves that the sequence (u_n) is bounded.

By Bolzano-Weierstrass theorem, (u_n) has a convergent subsequence.

Let l be the limit of that convergent subsequence. Then l is a subsequential limit of the sequence (u_n).

We now prove that the sequence (u_n) converges to l.

Let us choose $\epsilon > 0$. There exists a natural number k such that

$$| u_m - u_n | < \tfrac{\epsilon}{2} \text{ for all } m, n \geq k \ \ldots \ \ldots \ \text{(i)}$$

Since l is a subsequential limit of (u_n), there exists a natural number $q > k$ such that $| u_q - l | < \frac{\epsilon}{2}$.

Since $q > k$, from (i) $| u_q - u_n | < \frac{\epsilon}{2}$ for all $n \geq k$.

$$\begin{aligned} | u_n - l | &= | u_n - u_q | + | u_q - l | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } n \geq k. \end{aligned}$$

That is, $| u_n - l | < \epsilon$ for all $n \geq k$.

This implies $\lim u_n = l$. In other words, the sequence (u_n) is convergent and the theorem is done.

Definition. A subset S of the set $\mathbb{R}$ is said to be **complete** if every Cauchy sequence in S converges to a point in S.

The theorem says that the set $\mathbb{R}$ itself is complete. The set $\mathbb{Q}$ is not complete. We give here an example to establish it.

Let us consider the sequence (u_n) defined by $u_1 = 1.4, u_2 = 1.41, u_3 = 1.414, u_4 = 1.4142, \ldots$

u_n is the first $n+1$ digits in the decimal expansion of $\sqrt{2}$. This is to note that if $p, q \in \mathbb{N}$ and $p < q$, then the elements u_p, u_q of the sequence (u_n) agree for at least first $p+1$ places. So $|u_p - u_q| < \frac{1}{10^p}$.

Let us choose $\epsilon > 0$. Then there is a natural number k such that $0 < \frac{1}{10^k} < \epsilon$. Therefore for all $p, q > k$, $|u_p - u_q| < \frac{1}{10^p} < \frac{1}{10^k} < \epsilon$.

This shows that the sequence (u_n) is a Cauchy sequence in $\mathbb{Q}$. But the sequence (u_n) converges to $\sqrt{2}$, an irrational number.

Therefore the set $\mathbb{Q}$ is not complete.

Worked Examples (continued).

3. Prove that the sequence $(\frac{1}{n})$ is a Cauchy sequence.

Let $u_n = \frac{1}{n}$. Let $\epsilon > 0$. There is a natural number k such that $\frac{2}{k} < \epsilon$.

$$\text{Then } |u_m - u_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \le \frac{1}{m} + \frac{1}{n} < \epsilon \text{ if } m, n \ge k.$$

This proves that the sequence (u_n) is a Cauchy sequence.

4. Prove that the sequence $((-1)^n)$ is not a Cauchy sequence.

Let $u_n = (-1)^n$. Then

$$|u_m - u_n| = |(-1)^m - (-1)^n|.$$
$$|u_m - u_n| = 0 \text{ if } m \text{ and } n \text{ are both odd or both even,}$$
$$|u_m - u_n| = 2 \text{ if one of } m, n \text{ is odd and the other is even.}$$

Let us choose $\epsilon = \frac{1}{2}$. Then it is not possible to find a natural number k such that $|u_m - u_n| < \epsilon$ for all $m, n \ge k$.

Hence (u_n) is not a Cauchy sequence.

5. Prove that the sequence (u_n) where $u_1 = 0, u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \ge 1$, is a Cauchy sequence

$u_{n+2} - u_{n+1} = \frac{1}{2}(u_{n+1} + u_n) - u_{n+1} = -\frac{1}{2}(u_{n+1} - u_n)$

or, $|u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$.

Therefore $|u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n| = \frac{1}{2^2}|u_n - u_{n-1}|$
$= \cdots = \frac{1}{2^n}|u_2 - u_1| = \frac{1}{2^n}$.

Let $m > n$. Then $|u_m - u_n|$
$\le |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + \cdots + |u_{n+1} - u_n|$
$= (\frac{1}{2})^{m-2} + (\frac{1}{2})^{m-3} + \cdots + (\frac{1}{2})^{n-1}$

$= \frac{4}{2^n}[1 - (\frac{1}{2})^{m-n}] < \frac{4}{2^n}$.

Let $\epsilon > 0$. Then there exists a natural number k such that $\frac{4}{2^n} < \epsilon$ for all $n \ge k$. Hence $|u_m - u_n| < \epsilon$ for all $m, n \ge k$.

This proves that the sequence (u_n) is a Cauchy sequence.

6. Prove that the sequence (u_n) satisfying the condition $|u_{n+2} - u_{n+1}| \le c|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$, where $0 < c < 1$, is a Cauchy sequence.

$$\begin{aligned} |u_{n+2} - u_{n+1}| &\le c|u_{n+1} - u_n| \\ &\le c^2|u_n - u_{n-1}| \\ &\le \cdots \\ &\le c^n|u_2 - u_1|. \end{aligned}$$

Let $m > n$.

$$\begin{aligned} \text{Then } |u_m - u_n| &\le |u_m - u_{m-1}| + \cdots + |u_{n+1} - u_n| \\ &\le |u_2 - u_1|\{c^{m-2} + c^{m-3} + \cdots + c^{n-1}\} \\ &= |u_2 - u_1|\, c^{n-1}.\tfrac{1-c^{m-n}}{1-c} \\ &< \tfrac{c^{n-1}}{1-c}|u_2 - u_1|. \end{aligned}$$

Let $\epsilon > 0$. Since $0 < c < 1$, the sequence (c^{n-1}) is a convergent sequence. Therefore there exists a natural number k such that
$$\frac{c^{n-1}}{1-c}|u_2 - u_1| < \epsilon \text{ for all } n \ge k.$$

It follows that $|u_m - u_n| < \epsilon$ for all $m, n \ge k$ and this proves that the sequence (u_n) is a Cauchy sequence.

7. Prove that the sequence (u_n) defined by $u_1 = 3$ and $u_{n+1} = 3 + \frac{1}{u_n}$ for all $n \in \mathbb{N}$, is a Cauchy sequence. Find $\lim u_n$.

$|u_{n+2} - u_{n+1}| = \left|\frac{1}{u_{n+1}} - \frac{1}{u_n}\right| = \left|\frac{u_{n+1} - u_n}{u_{n+1}u_n}\right|$.

$u_1 = 3, u_n > 3$ for all $n > 1$.

Therefore $|u_{n+2} - u_{n+1}| < \frac{1}{9}|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$. By the previous example, (u_n) is a Cauchy sequence and therefore it is a convergent sequence.

Let l be the limit of the sequence (u_n). We have $u_{n+1} = 3 + \frac{1}{u_n}$ for all $n \in \mathbb{N}$. Proceeding to limit as $n \to \infty$, we have $l = 3 + \frac{1}{l}$.

This gives $l = \frac{1}{2}(3 + \sqrt{13})$. [$l$ is not negative, since each element of the sequence is positive.]

Note. The limit of the sequence (u_n) is the value of the *infinite continued fraction* $3 + \frac{1}{3+}\frac{1}{3+}\frac{1}{3+}\cdots$. The successive *convergents* of the continued fraction are $3, 3 + \frac{1}{3}, 3 + \frac{1}{3+\frac{1}{3}}, \ldots$

5.15. Cauchy's theorems on limits.

Theorem 5.15.1. Let (u_n) be a sequence and $\lim u_n = l$. Then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

Proof. **Case 1.** $l = 0$.

Since (u_n) is a convergent sequence, it is bounded. Therefore there exists a positive real number B such that $| u_n |< B$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\lim u_n = 0$, there exists a natural number k_1 such that $| u_n |< \frac{\epsilon}{2}$ for all $n \geq k_1$.

$$|\frac{u_1+u_2+\cdots+u_n}{n}| \leq |\frac{u_1+u_2+\cdots+u_{k_1-1}}{n}| + |\frac{u_{k_1}+u_{k_1+1}+\cdots+u_n}{n}|$$
$$\leq |\frac{|u_1|+|u_2|+\cdots+|u_{k_1-1}|}{n}| + |\frac{|u_{k_1}|+|u_{k_1+1}|+\cdots+|u_n|}{n}|$$
$$< \frac{B(k_1-1)}{n} + \frac{n-k_1+1}{n}.\frac{\epsilon}{2} \text{ for all } n \geq k_1$$
$$< \frac{Bk_1}{n} + \frac{\epsilon}{2} \text{ for all } n \geq k_1.$$

Since $\lim \frac{1}{n} = 0$, there exists a natural number k_2 such that $\frac{Bk_1}{n} < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|\frac{u_1+u_2+\cdots+u_n}{n}|< \epsilon$ for all $n \geq k$.

This proves that $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$.

Case 2. $l \neq 0$.

Let $v_n = u_n - l$. Then $\lim v_n = 0$.

Now $\frac{u_1+u_2+\cdots+u_n}{n} - l = \frac{v_1+v_2+\cdots+v_n}{n}$.

By case 1, $\lim \frac{v_1+v_2+\cdots+v_n}{n} = 0$. Therefore $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

This completes the proof.

Note. The converse of the theorem is not true.

Let us consider the sequence (u_n), where $u_n = (-1)^n$.

Then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$, but the sequence (u_n) is not convergent.

Corollary. If $\lim u_n = l$ where $u_n > 0$ for all n and $l \neq 0$, then $\lim \sqrt[n]{u_1 u_2 \ldots u_n} = l$.

Since each u_n is positive and $\lim u_n = l > 0$, the sequence $(\log u_n)$ converges to $\log l$, by the Corollary of 4 of Art 5.8.

Therefore $\lim \frac{\log u_1+\log u_2+\cdots+\log u_n}{n} = \log l$.

or, $\lim \log \sqrt[n]{(u_1 u_2 \ldots u_n)} = \log l$.

It follows that, $\lim \sqrt[n]{(u_1 u_2 \ldots u_n)} = l$.

Worked Examples.

1. Prove that $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

Let $u_n = \frac{1}{n}$. Then $\lim u_n = 0$.

By Cauchy's theorem, $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

2. Prove that $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Let $u_n = \sqrt[n]{n}$. Then $\lim u_n = 1$.

By Cauchy's theorem, $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Theorem 5.15.2. Let (u_n) be a real sequence such that $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l(\neq 0)$. Then $\lim \sqrt[n]{u_n} = l$.

Proof. Let $v_1 = u_1, v_2 = \frac{u_2}{u_1}, v_3 = \frac{u_3}{u_2}, \cdots, v_n = \frac{u_n}{u_{n-1}}, \cdots$

Then $v_n > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = l > 0$

This implies $\lim \log v_n = \log l$.

By the first theorem, $\lim \frac{\log v_1+\log v_2+\cdots+\log v_n}{n} = \log l$.

or, $\lim \log \sqrt[n]{(v_1 v_2 \ldots v_n)} = \log l$.

It follows that $\lim \sqrt[n]{(v_1 v_2 \ldots v_n)} = l$. That is, $\lim \sqrt[n]{u_n} = l$.

Theorem 5.15.3. Let (u_n) be a real sequence such that $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l$ (finite or infinite). Then $\lim \sqrt[n]{u_n} = l$.

Proof. **Case 1.** $0 < l < \infty$.

Let us choose $\epsilon > 0$ such that $l - \epsilon > 0$. Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that $l - \frac{\epsilon}{2} < \frac{u_{n+1}}{u_n} < l + \frac{\epsilon}{2}$ for all $n \geq k$.

Then $l - \frac{\epsilon}{2} < \frac{u_{k+1}}{u_k} < l + \frac{\epsilon}{2}$

$l - \frac{\epsilon}{2} < \frac{u_{k+2}}{u_k} < l + \frac{\epsilon}{2}$

$\ldots \quad \ldots$

$l - \frac{\epsilon}{2} < \frac{u_n}{u_{n+1}} < l + \frac{\epsilon}{2}$.

We have $(l - \frac{\epsilon}{2})^{n-k} < \frac{u_n}{u_k} < (l + \frac{\epsilon}{2})^{n-k}$ for all $n > k$

or, $(l - \frac{\epsilon}{2})^n.B < u_n < A.(l + \frac{\epsilon}{2})^n$, where $A = \frac{u_k}{(l+\frac{\epsilon}{2})^k} > 0, B = \frac{u_k}{(l-\frac{\epsilon}{2})^k} > 0$.

or, $(l - \frac{\epsilon}{2})B^{\frac{1}{n}} < u_n^{\frac{1}{n}} < A^{\frac{1}{n}}(l + \frac{\epsilon}{2})$.

Since $A > 0, \lim A^{\frac{1}{n}} = 1$. Since $B > 0, \lim B^{\frac{1}{n}} = 1$.

Since $\lim A^{\frac{1}{n}}(l + \frac{\epsilon}{2}) = l + \frac{\epsilon}{2}$, there exists a natural number k_2 such that $A^{\frac{1}{n}}(l + \frac{\epsilon}{2}) < l + \epsilon$ for all $n \geq k_2$.

Since $\lim B^{\frac{1}{n}}(l - \frac{\epsilon}{2}) = l - \frac{\epsilon}{2}$, there exists a natural number k_3 such that $B^{\frac{1}{n}}(l - \frac{\epsilon}{2}) > l - \epsilon$ for all $n \geq k_3$.

Let $k_0 = \max\{k_1, k_2, k_3\}$. Then $l - \epsilon < u_n^{\frac{1}{n}} < l + \epsilon$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = l$.

Case 2. $l = 0$.

Let $\epsilon > 0$. There exists a natural number k such that

$$0 < \frac{u_{n+1}}{u_n} < \frac{\epsilon}{2} \text{ for all } n \geq k.$$

Therefore $0 < \frac{u_{k+1}}{u_k} < \frac{\epsilon}{2}, 0 < \frac{u_{k+2}}{u_{k+1}} < \frac{\epsilon}{2}, \cdots\cdots, 0 < \frac{u_n}{u_{n-1}} < \frac{\epsilon}{2}$.

We have $0 < \frac{u_n}{u_k} < (\frac{\epsilon}{2})^{n-k}$ for all $n > k$

or, $0 < u_n < \frac{u_k}{(\frac{\epsilon}{2})^k}.(\frac{\epsilon}{2})^n$

or, $0 < u_n < A(\frac{\epsilon}{2})^n$ where $A = u_k(\frac{2}{\epsilon})^k > 0$

or, $0 < u_n^{\frac{1}{n}} < A^{\frac{1}{n}}\frac{\epsilon}{2}$.

Since $A > 0, \lim A^{\frac{1}{n}} = 1$.

Since $\lim A^{\frac{1}{n}}\frac{\epsilon}{2} = \frac{\epsilon}{2}$, there exists a natural number k_1 such that $A^{\frac{1}{n}}\frac{\epsilon}{2} < \epsilon$ for all $n \geq k_1$.

Let $k_0 = \max\{k, k_1\}$. Then $0 < u_n^{\frac{1}{n}} < \epsilon$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = 0$.

Case 3. $\lim \frac{u_{n+1}}{u_n} = \infty$.

Let us choose $G > 0$. There exists a natural number k such that $\frac{u_{n+1}}{u_n} > G + 1$ for all $n \geq k$.

Therefore $\frac{u_{k+1}}{u_k} > G+1, \frac{u_{k+2}}{u_{k+1}} > G+1, \cdots, \frac{u_n}{u_{n-1}} > G+1$.

We have $\frac{u_n}{u_k} > (G+1)^{n-k}$ for all $n > k$

or, $u_n > \mu(G+1)^n$ where $\mu = \frac{u_k}{(G+1)^k} > 0$

or, $u_n^{\frac{1}{n}} > \mu^{\frac{1}{n}}(G+1)$.

Since $\mu > 0, \lim \mu^{\frac{1}{n}} = 1$.

Since $\lim \mu^{\frac{1}{n}}(G+1) = G+1$, there exists a natural number k_1 such that $\mu^{\frac{1}{n}}(G+1) > G$ for all $n \geq k_1$.

Let $k_0 = \max\{k, k_1\}$. Then $u_n^{\frac{1}{n}} > G$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = \infty$.

This completes the proof.

Note. The converse of the theorem is not true. To establish this, let us consider the sequence (u_n), where $u_n = \frac{3+(-1)^n}{2}$ for all $n \in \mathbb{N}$.
The sequence is $(1, 2, 1, 2, 1, 2, \ldots\ldots)$.
Here $\lim \sqrt[n]{u_n} = 1$, since $\lim(u_{2n})^{\frac{1}{2n}} = 1$ and $\lim(u_{2n-1})^{\frac{1}{2n-1}} = 1$.
But $\lim \frac{u_{n+1}}{u_n}$ does not exist.

Theorem 5.15.4. Let (u_n) be a real sequence such that $u_n > 0$ for all $n \in \mathbb{N}$. Then

$$\underline{\lim}\frac{u_{n+1}}{u_n} \leq \underline{\lim}\sqrt[n]{u_n} \leq \overline{\lim}\sqrt[n]{u_n} \leq \overline{\lim}\frac{u_{n+1}}{u_n}.$$

Proof. Let $\underline{\lim}\frac{u_{n+1}}{u_n} = \lambda_*, \overline{\lim}\frac{u_{n+1}}{u_n} = \lambda^*, \underline{\lim}\sqrt[n]{u_n} = \mu_*, \overline{\lim}\sqrt[n]{u_n} = \mu^*$.

We first prove $\mu^* \leq \lambda^*$

Case 1. Let $\lambda^* = \infty$. Then $\mu^* \leq \lambda^*$ trivially.

Case 2. Let λ^* be finite.

Let us choose $\epsilon > 0$. Since $\overline{\lim}\frac{u_{n+1}}{u_n} = \lambda^*$, there exists a natural number k such that $\frac{u_{n+1}}{u_n} < \lambda^* + \epsilon$ for all $n \geq k$.

Then $\frac{u_{k+1}}{u_k} < \lambda^* + \epsilon, \frac{u_{k+2}}{u_{k+1}} < \lambda^* + \epsilon, \cdots, \frac{u_n}{u_{n-1}} < \lambda^* + \epsilon$

So $u_n < u_k(\lambda^* + \epsilon)^{n-k}$ for all $n > k$.

Hence for all $n > k, u_n < A(\lambda^* + \epsilon)^n$ where $A = \frac{u_k}{(\lambda^*+\epsilon)^k} > 0$.

Therefore $\sqrt[n]{u_n} < A^{1/n}(\lambda^* + \epsilon)$ for all $n > k$.

Consequently, $\overline{\lim}\sqrt[n]{u_n} \leq \overline{\lim} A^{1/n}(\lambda^* + \epsilon)$
$= \lambda^* + \epsilon$ since $\lim A^{1/n} = 1$.

Since ϵ is arbitrary, $\overline{\lim}\sqrt[n]{u_n} \leq \lambda^*$, i.e., $\mu^* \leq \lambda^*$.

In a similar manner we can prove $\lambda_* \leq \mu_*$.

Also the inequality $\mu_* \leq \mu^*$ follows from the property of the limit inferior and the limit superior of a sequence.

This completes the proof.

Note. If $u_n > 0$ for all n and $\lim \frac{u_{n+1}}{u_n}$ exists, then it follows from the theorem that $\lim \sqrt[n]{u_n}$ also exists.

Worked Examples (continued).

3. Prove that $\lim \sqrt[n]{n} = 1$.

Let $u_n = n$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = 1 > 0$.
It follows that from the theorem that $\lim \sqrt[n]{n} = 1$.

4. Prove that $\lim \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

Let $u_n = \frac{n!}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \frac{1}{e} > 0$.

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{1}{e}$, i.e., $\lim \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

5. Prove that $\lim \frac{\{(n+1)(n+2)\cdots(2n)\}^{1/n}}{n} = \frac{4}{e}$.

Let $u_n = \frac{(n+1)(n+2)\cdots 2n}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \lim \frac{2(2n+1)}{n+1}.\frac{1}{(1+\frac{1}{n})^n} = \frac{4}{e} > 0$.

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{4}{e}$.

6. SERIES

6.1. Infinite Series.

Let (u_n) be a sequence. Then the sequence (s_n) defined by

$$s_1 = u_1, s_2 = u_1 + u_2, s_3 = u_1 + u_2 + u_3, \cdots \cdots$$

is represented by the symbol $u_1 + u_2 + u_3 + \cdots \cdots$, which is said to be an *infinite series* (or a *series*) generated by the sequence (u_n).

The series is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or by Σu_n. u_n is said to be the *nth term* of the series.

The elements of the sequence (s_n) are called the partial sums of the series Σu_n and the sequence (s_n) is called the *sequence of partial sums* of the series Σu_n.

If (u_n) be a real sequence, then Σu_n is a series of real numbers.

We shall be mainly concerned with the series of real numbers.

The infinite series Σu_n is said to be *convergent* or *divergent* according as the sequence (s_n) is convergent or divergent.

If the sequence (s_n) is convergent with $\lim s_n = s$, then s is said to be the *sum* of the series Σu_n. If, however, $\lim s_n = \infty$ (or $-\infty$), then the series Σu_n is said to *diverge* to ∞ (or $-\infty$).

Examples.

1. Let us consider the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \cdots$

Let the series be $\sum\limits_{n=1}^{\infty} u_n$. Then $u_n = \frac{1}{n(n+1)}$.

Let $s_n = u_1 + u_2 + \cdots + u_n$.

$$\begin{aligned} \text{Then } s_n &= \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

and $\lim s_n = 1$. Hence the series Σu_n is convergent and the sum of the series is 1.

2. Let us consider the series $1 + 2 + 3 + \cdots \cdots$

Let $s_n = 1+2+3+\cdots+n$. Then $s_n = \frac{n(n+1)}{2}$ and $\lim s_n = \infty$.
Hence the series is divergent.

3. Let us consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots$

Let $s_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$.
Then $s_n = 2(1-\frac{1}{2^n}) = 2-\frac{1}{2^{n-1}}$ and $\lim s_n = 2$, since $\lim(\frac{1}{2})^{n-1} = 0$.
Therefore the series is convergent and the sum of the series is 2.

4. Let us consider the series $1-1+1-1+\cdots\cdots$

Let $s_n = 1-1+1-1+\cdots+(-1)^{n-1}$.
Then $s_n = 0$ if n be even,
$= 1$ if n be odd.
The sequence (s_n) is divergent. Therefore the series is divergent.

5. **Geometric series.**

A. Let us consider the series $1+a+a^2+\cdots$ where $|a|<1$.

Let $s_n = 1+a+a^2+\cdots+a^{n-1}$. Then $s_n = \frac{1-a^n}{1-a} = \frac{1}{1-a} - \frac{a^n}{1-a}$.
$\lim s_n = \frac{1}{1-a}$, since $\lim a^n = 0$.
Therefore the series is convergent and the sum of the series is $\frac{1}{1-a}$.

B. Let us consider the series $1+a+a^2+\cdots$ where $|a|\geq 1$.
Let $s_n = 1+a+a^2+\cdots+a^{n-1}$.

Case 1. $a=1$. In this case $s_n = n$ and $\lim s_n = \infty$.
Therefore the series is divergent.

Case 2. $a>1$. In this case $s_n = \frac{a^n-1}{a-1}$ and $\lim s_n = \infty$ since $\lim a^n = \infty$ in this case.
Therefore the series is divergent.

Case 3. $a=-1$. In this case $s_n = 1$ if n be odd,
$= 0$ if n be even.
The sequence (s_n) is divergent. Therefore the series is divergent.

Case 4. $a<-1$. In this case the sequence (s_n) is divergent and therefore the series is divergent.

From (A) and (B), the geometric series $1+a+a^2+\cdots\cdots$ is convergent if $|a|<1$, and divergent if $|a|\geq 1$.

6. **Harmonic series.**

$1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the series. Then $u_n = \frac{1}{n}$.

$$
\begin{aligned}
\text{Let } s_n &= u_1+u_2+\cdots+u_n.\\
\text{Then } s_2 &= 1+\tfrac{1}{2}\\
s_4 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4} > 1+\tfrac{1}{2}+(\tfrac{1}{4}+\tfrac{1}{4}) = 1+2.\tfrac{1}{2}\\
s_8 &= 1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})\\
&> 1+\tfrac{1}{2}+(\tfrac{1}{4}+\tfrac{1}{4})+(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}) = 1+3.\tfrac{1}{2}\\
s_{16} &> 1+4.\tfrac{1}{2}\\
&\cdots \quad \cdots\\
s_{2^n} &> 1+n.\tfrac{1}{2}.
\end{aligned}
$$

Therefore $\lim_{n\to\infty} s_{2^n} = \infty$.

The sequence (s_n) is a monotone increasing sequence, since $s_{n+1} - s_n = a_{n+1} > 0$ for all $n \in \mathbb{N}$. Since the subsequence (s_{2^n}) diverges to ∞, the sequence (s_n) is unbounded above and therefore the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Theorem 6.1.1. Let m be a natural number. Then the two series $u_1+u_2+u_3+\cdots\cdots$ and $u_{m+1}+u_{m+2}+u_{m+3}+\cdots\cdots$ converge or diverge together.

Proof. Let $s_n = u_1+u_2+\cdots+u_n, t_n = u_{m+1}+u_{m+2}+\cdots+u_{m+n}$.
Then $t_n = s_{m+n} - s_m$, where s_m is a fixed number.
If the sequence (s_n) converges, then the sequence (t_n) converges and conversely.
If the sequence (s_n) diverges, then the sequence (t_n) diverges and conversely.
Therefore both the sequences (s_n) and (t_n) and consequently the series Σu_n and Σu_{m+n} converge or diverge together.

Note. The theorem states that we can remove from the beginning a finite number of terms from a given series or we can add to the beginning a finite number of terms to a given series, without changing its behaviour regarding convergence or divergence.

Theorem 6.1.2. If Σu_n and Σv_n be two convergent series having the sums s and t respectively, then

(i) the series $\Sigma(u_n+v_n)$ converges to the sum $s+t$;

(ii) the series $\Sigma k u_n$, k being a real number, converges to the sum ks.

The proof is immediate.

Theorem 6.1.3. Cauchy's principle of convergence.

A necessary and sufficient condition for the convergence of a series Σu_n is that corresponding to a pre-assigned positive ϵ, there exists a natural number m such that

$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n$.

Let Σu_n be convergent. Then the sequence (s_n) is convergent. By Cauchy's principle of convergence for the sequence, corresponding to a pre-assigned positive ϵ there exists a natural number m such that

$|s_{n+p} - s_n| < \epsilon$ for all $n \geq m$ and for every natural number p.

or, $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

Conversely, let us assume that for a pre-assigned positive ϵ there exists a natural number m such that

$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

Then $|s_{n+p} - s_n| < \epsilon$ for all $n \geq m$ and for every natural number p.

This implies that the sequence (s_n) is convergent by Cauchy's principle of convergence. Therefore Σu_n is convergent.

This completes the proof.

Theorem 6.1.4. A necessary condition for the convergence of a series Σu_n is $\lim u_n = 0$.

Proof. Let Σu_n be convergent. Then for a pre-assigned positive ϵ there exists a natural number m such that

$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

Taking $p = 1$, $|u_{n+1}| < \epsilon$ for all $n \geq m$. This implies $\lim u_n = 0$.

Note. The converse of the theorem is not true.

That is, $\lim u_n = 0$ does not necessarily imply convergence of the series Σu_n. Because the sufficient condition for the convergence of the series Σu_n states that for a chosen positive ϵ there must exist a natural number m such that

$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \cdots$.

Therefore the sum of p consecutive terms of the series must be less than ϵ whatever natural number p may be. The condition must be satisfied for all p and not for only a particular p.

Let us consider the series Σu_n, where $u_n = \frac{1}{n}$. Here $\lim u_n = 0$. But Σu_n is a divergent series.

Here $|s_{n+p} - s_n| = |\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}|$.

If we take $p = n$,
$$|s_{n+p} - s_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

Therefore $|s_{n+p} - s_n|$ cannot be made less than a chosen positive $\epsilon < \frac{1}{2}$ for every natural number p.

Worked Examples.

1. Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ is convergent.

Let the series be $\sum\limits_{n=1}^{\infty} u_n$. Then $u_n = (-1)^{n+1}\frac{1}{n}$.

Let $s_n = u_1 + u_2 + \cdots + u_n$. Then

$$\begin{aligned} |s_{n+p} - s_n| &= \left|\frac{1}{n+1} - \frac{1}{n+2} + \cdots + (-1)^{p-1}\frac{1}{n+p}\right| \\ &= \left|\frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \cdots\right| \\ &< \frac{1}{n+1}. \end{aligned}$$

Let $\epsilon > 0$. Then $|s_{n+p} - s_n| < \epsilon$ holds if $n > \frac{1}{\epsilon} - 1$.

Let $m = [\frac{1}{\epsilon} - 1] + 2$. Then m is a natural number and $|s_{n+p} - s_n| < \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \cdots$

This proves that the sequence (s_n) is convergent and consequently, the series Σu_n is convergent.

2. Prove that the series $\sum\limits_{n=1}^{\infty} u_n$ where $u_n = \frac{n}{n+1}$, is divergent.

Here $\lim u_n = 1$. Since $\lim u_n$ is not 0, Σu_n is divergent because a necessary condition for the convergence of a series Σu_n is $\lim u_n = 0$.

6.2. Series of positive terms.

A series Σu_n is said to be a *series of positive terms* if u_n is a positive real number for all $n \in \mathbb{N}$.

Theorem 6.2.1. A series of positive real numbers Σu_n is convergent if and only if the sequence (s_n) of partial sums is bounded above.

Proof. $s_n = u_1 + u_2 + \cdots + u_n$. Then $s_{n+1} - s_n = u_{n+1} > 0$ for all $n \in \mathbb{N}$.

Hence the sequence (s_n) is a monotone increasing sqeuence. Therefore (s_n) is convergent if and only if it is bounded above.

Consequently, the series Σu_n is convergent if and only if the sequence (s_n) is bounded above.

Note. If not bounded above, the sequence (s_n) being a monotone increasing sequence, diverges to ∞. In this case the series diverges to ∞.

Therefore a series of positive real numbers either converges to a real number, or diverges to ∞.

Introduction and removal of brackets.

Let Σu_n be a series of positive real numbers. Let the terms of the series be arranged in groups without changing the order of the terms. Let us denote the nth group by v_n. Then a new series Σv_n is obtained.

Example.

Let $\Sigma u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \cdots$

Let us introduce brackets and the series takes the form
$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots$

If the new series be Σv_n, then
$v_1 = 1, v_2 = \frac{1}{2}, v_3 = \frac{1}{3} + \frac{1}{4}, v_4 = \frac{1}{5} + \cdots + \frac{1}{8}, v_5 = \frac{1}{9} + \cdots + \frac{1}{16}, \cdots$

Σv_n is obtained from Σu_n by introduction of brackets and Σu_n is obtained from Σv_n by removal of brackets.

Theorem 6.2.2. Let Σu_n be a series of positive real numbers and Σv_n is obtained from Σu_n by grouping its terms. Then

(i) if Σu_n converges to the sum s, so does Σv_n ;

(ii) if Σv_n converges to the sum t, so does Σu_n.

Proof. Let $v_1 = u_1 + u_2 + \cdots + u_{r_1}, v_2 = u_{r_1+1} + \cdots + u_{r_2}, v_3 = u_{r_2+1} + \cdots + u_{r_3}, \cdots \cdots$

Then (r_n) is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n,\ \ t_n = v_1 + v_2 + \cdots + v_n$.

Then $t_n = u_1 + u_2 + \cdots + u_{r_n} = s_{r_n}$.

Let Σu_n be convergent and the sum of the series be s. Then $\lim s_n = s$. The sequence (s_{r_n}) being a subsequence of the convergent sequence (s_n), is convergent and $\lim s_{r_n} = s$. That is, $\lim t_n = s$.

This proves that the series Σv_n is convergent and the sum of the series is also s.

Let Σv_n be convergent and the sum of the series be t. Then $\lim t_n = t$. That is, $\lim s_{r_n} = t$.

(s_{r_n}) is a convergent subsequence of the monotone increasing sequence (s_n). By Theorem 5.11.4, the sequence (s_n) is convergent and $\lim s_n = t$.

This proves that the series Σu_n is convergent and the sum of the series is also t. This completes the proof.

Note. The theorem does not hold if Σu_n be a series of arbitrary terms.

Let us consider the series

$1 - 1 + 1 - 1 + \cdots \cdots \cdots$ ...(A)

Introducing brackets we get the series

$(1 - 1) + (1 - 1) + \cdots \cdots \cdots$ ...(B)

Introducing brackets in another manner we get the series

$1 - (1 - 1) - (1 - 1) - \cdots \cdots \cdots$ ...(C)

The series (A) is divergent but the series (B) converges to 0 and the series (C) converges to 1.

Re-arrangement of terms.

Let Σu_n be a given series. If a new series Σv_n is obtained by using each term of Σu_n exactly once, the order of the terms being disturbed, then Σv_n is called a re-arrangement of Σu_n.

If $f : \mathbb{N} \to \mathbb{N}$ be a bijective mapping, $\Sigma u_{f(n)}$ is a re-arrangement of Σu_n and conversely if Σv_n be a re-arrangement of the series Σu_n then $v_n = u_{f(n)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$.

For example, let $f(n) \;=\; n + 1$ if n be odd .
$\qquad\qquad\qquad\quad\;\; = \; n - 1$ if n be even.

$f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 3, \cdots \cdots$

$\Sigma u_{f(n)} = u_2 + u_1 + u_4 + u_3 + \cdots \cdots$ is a re-arrangement of Σu_n.

Theorem 6.2.3. Let Σu_n be a convergent series of positive real numbers.Then any re-arrangement of Σu_n is convergent and the sum remains unaltered.

Proof. Let Σu_n converge to s and Σv_n be a re-arrangement of Σu_n. Then $v_n = u_{f(n)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$.

Let $s_n = u_1 + u_2 + \cdots + u_n,\ \ t_n = v_1 + v_2 + \cdots + v_n$.

Since $u_n > 0$, the sequence (s_n) is a monotone increasing sequence. As Σu_n converges to s, $\lim s_n = s$. Therefore the sequence (s_n) is bounded above and $s_n \leq s$ for all $n \in \mathbb{N}$.

$$\begin{aligned} t_n &= v_1 + v_2 + \cdots + v_n \\ &= u_{f(1)} + u_{f(2)} + \cdots + u_{f(n)} \\ &\leq u_1 + u_2 + \cdots + u_{m(n)}, \text{ where } m(n) = \max\{f(1), \ldots, f(n)\}. \end{aligned}$$

But $u_1 + u_2 + \cdots + u_{m(n)} = s_{m(n)} \leq s$.

Thus the sequence (t_n) is bounded above and being a monotone increasing sequence, it is convergent. Let $\lim t_n = t$. Then $t \leq s$.

$$\begin{aligned} s_n &= u_1 + u_2 + \cdots + u_n \\ &= v_{f^{-1}(1)} + v_{f^{-1}(2)} + \cdots + v_{f^{-1}(n)} \\ &\leq v_1 + v_2 + \cdots + v_{k(n)}, \text{ where } k(n) = \max\{f^{-1}(1), \ldots, f^{-1}(n)\}. \end{aligned}$$

But $v_1 + v_2 + \cdots + v_{k(n)} = t_{k(n)} \leq t$.

$s_n \leq t \Rightarrow \lim s_n \leq t$, i.e., $s \leq t$

It follows that $s = t$.

This proves that the series Σv_n is convergent and the sum of the series is also s.

6.3. Tests for convergence of a series of positive terms.

The convergence or divergence of a particular series is decided by examining the sequence of partial sums of the series. In most cases the expression for s_n (the nth partial sum) becomes not so nice as can be easily handled to determine its nature in a straightforward manner. Some other elegent methods will be applied to the series that will decide the convergence of the series without prior knowledge of the nature of the sequence (s_n). These methods, called 'tests for convergence', will be discussed here.

Theorem 6.3.1. Comparison test [First type].

A. Let Σu_n and Σv_n be two series of positive real numbers and there is a natural number m such that $u_n \leq kv_n$ for all $n \geq m, k$ being a fixed positive number.

Then (i) Σu_n is convergent if Σv_n is convergent,

(ii) Σv_n is divergent if Σu_n is divergent.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n,\ t_n = v_1 + v_2 + \cdots + v_n$.

$$\begin{aligned} \text{Then } s_n - s_m &= u_{m+1} + u_{m+2} + \cdots + u_n \\ &\leq k(v_{m+1} + v_{m+2} + \cdots + v_n) \\ &= k(t_n - t_m) \end{aligned}$$

or, $s_n \leq kt_n + h$ where $h = s_m - kt_m$, a finite number.

(i) Let Σv_n be convergent. Then the sequence (t_n) is bounded.
Let B be an upper bound. Then $t_n < B$ for all $n \in \mathbb{N}$.
Therefore $s_n < kB + h$ for all $n \geq m$.
This shows that the sequence (s_n) is bounded above. (s_n) being a monotone increasing sequence bounded above, is convergent.
Therefore Σu_n is convergent.

(ii) Let Σu_n is divergent. Then the sequence (s_n) is not bounded above.
Since $s_n \leq kt_n + h$, the sequence (t_n) is not bounded above. Therefore the series Σv_n is divergent.

B. Limit form.

Let Σu_n and Σv_n be two series of positive real numbers and $\lim \frac{u_n}{v_n} = l$ where l is a *non-zero* finite number.
Then the two series Σu_n and Σv_n converge or diverge together.

Proof. $l > 0$. Let us choose a positive ϵ such that $l - \epsilon > 0$. There a natural number m such that $l - \epsilon < \frac{u_n}{v_n} < l + \epsilon$ for all $n \geq m$.
Therefore $u_n < kv_n$ for all $n \geq m$ where $k = l + \epsilon > 0$ ... ... (i)

and $v_n < k'u_n$ for all $n \geq m$ where $k' = \frac{1}{l-\epsilon} > 0$ ... ... (ii)

By comparison test A, it follows from (i) that Σu_n is convergent if Σv_n is convergent and Σv_n is divergent if Σu_n is divergent.

By comparison test A, it follows from (ii) that Σv_n is convergent if Σu_n is convergent and Σu_n is divergent if Σv_n is divergent.

Therefore the two series Σu_n and Σv_n converge or diverge together.

Note. If $\lim \frac{u_n}{v_n} = 0$, then for a pre-assigned positive number ϵ there exists a natural number m such that $0 < \frac{u_n}{v_n} < \epsilon$ for all $n \geq m$.
Therefore Σu_n is convergent if Σv_n is convergent.

If $\lim \frac{u_n}{v_n} = \infty$, then for a pre-assigned positive number G there exists a natural number m such that $\frac{u_n}{v_n} > G$ for all $n \geq m$.
Therefore Σu_n is divergent if Σv_n is divergent.

In order to make use of the Comparison test we need to have a collection of series of known behaviour. The series $\Sigma \frac{1}{n^p}$ discussed in the following theorem will be an addition to the collection.

Theorem 6.3.2. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \cdots$ converges for $p > 1$ and diverges for $p \leq 1$.

Proof. **Case 1.** $p > 1$. Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{1}{n^p}$.
Let Σv_n be obtained from Σu_n by grouping the terms as
$1 + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}) + (\frac{1}{8^p} + \cdots + \frac{1}{15^p}) + \cdots$

$$\begin{aligned} \text{Then } v_1 = 1, v_2 &= \tfrac{1}{2^p} + \tfrac{1}{3^p} < \tfrac{1}{2^p} + \tfrac{1}{2^p} = \tfrac{1}{2^{p-1}}, \\ v_3 &= \tfrac{1}{4^p} + \cdots + \tfrac{1}{7^p} < \tfrac{4}{4^p} = \tfrac{1}{2^{2(p-1)}}, \\ v_4 &= \tfrac{1}{8^p} + \cdots + \tfrac{1}{15^p} < \tfrac{8}{8^p} = \tfrac{1}{2^{3(p-1)}}, \\ \cdots & \quad \cdots \quad \cdots \end{aligned}$$

Let $w_n = \{\frac{1}{2^{p-1}}\}^{n-1}$. Then $v_n < w_n$ for all $n \geq 2$.
But Σw_n is a geometric series of common ratio $\frac{1}{2^{p-1}}$.
Since $p > 1, 0 < \frac{1}{2^{p-1}} < 1$ and hence Σw_n is convergent.

Therefore Σv_n is convergent by Comparison test.

Since the series Σu_n is a series of positive terms and Σv_n is obtained from Σu_n by introduction of brackets, Σu_n is convergent.

Case 2. $p = 1$. In this case the series is $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots$
Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

$$\begin{aligned} \text{Then } s_{2n} - s_n &= \tfrac{1}{n+1} + \tfrac{1}{n+2} + \cdots + \tfrac{1}{2n} \\ &> \tfrac{1}{2n} + \tfrac{1}{2n} + \cdots + \tfrac{1}{2n} = \tfrac{1}{2}. \end{aligned}$$

This shows that the sequence (s_n) is not a Cauchy sequence and therefore is not convergent. Hence the series $\Sigma\frac{1}{n}$ is not convergent.

Case 3. $0 < p < 1$. Then $\frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}, \cdots\cdots$

Therefore $\frac{1}{n^p} > \frac{1}{n}$ for all $n \geq 2$.

But $\Sigma\frac{1}{n}$ is divergent. Therefore $\Sigma\frac{1}{n^p}$ is divergent by Comparison test.

Case 4. $p \leq 0$. Then $\lim \frac{1}{n^p} \neq 0$ and therefore $\Sigma\frac{1}{n^p}$ is not convergent.

This completes the proof.

Worked Examples.

1. Test the convergence of the series $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots\cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{n+2}{2(n+1)^2}$.

Let $v_n = \frac{1}{n}$. Then $\lim \frac{u_n}{v_n} = \lim \frac{n(n+2)}{2(n+1)^2} = \frac{1}{2}$.

Since Σv_n is divergent, Σu_n is divergent by Comparison test.

2. Test the convergence of the series $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \cdots\cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{1}{n(n+1)^2}$.

Let $v_n = \frac{1}{n^3}$. Then $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1$.

But Σv_n is convergent. Therefore Σu_n is convergent by Comparison test.

3. Test the convergence of the series Σu_n where $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$. Let $v_n = \frac{1}{n^2}$.

Then $\lim \frac{u_n}{v_n} = \lim \frac{2n^2}{n^2\{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\}} = 1$.

Since Σv_n is convergent, Σu_n is convergent by Comparison test.

Theorem 6.3.3. Comparison test [Second type].

Let Σu_n and Σv_n be two series of positive real numbers and there is natural number m such that

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n} \text{ for all } n \geq m.$$

Then (i) Σu_n is convergent if Σv_n is convergent,

(ii) Σv_n is divergent if Σu_n is divergent.

Proof. $\frac{u_{m+1}}{u_m} \leq \frac{v_{m+1}}{v_m}, \frac{u_{m+2}}{u_{m+1}} \leq \frac{v_{m+2}}{v_{m+1}}, \cdots, \frac{u_n}{u_{n-1}} \leq \frac{v_n}{v_{n-1}}$, where $n > m$.

Therefore $\frac{u_n}{u_m} \leq \frac{v_n}{v_m}$ for all $n > m$

or, $u_n \leq \frac{u_m}{v_m} v_n$ for all $n > m$

or, $u_n \leq k v_n$ for all $n > m$ and $k(= \frac{u_m}{v_m})$ is a positive number.

By Comparison test (first type), Σu_n is convergent if Σv_n is convergent and Σv_n is divergent if Σu_n is divergent.

Theorem 6.3.4. D'Alembert's ratio test.

Let Σu_n be a series of positive real numbers and let $\lim \frac{u_{n+1}}{u_n} = l$.

Then Σu_n is convergent if $l < 1$, Σu_n is divergent if $l > 1$.

Proof. **Case 1.** $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number m such that

$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon$ for all $n \geq m$.

Let $l + \epsilon = r$. Then $0 < r < 1$.

We have $\frac{u_{m+1}}{u_m} < r, \frac{u_{m+2}}{u_{m+1}} < r, \cdots\cdots, \frac{u_n}{u_{n-1}} < r$, where $n > m$.

Consequently, $\frac{u_n}{u_m} < r^{n-m}$ for all $n > m$

or, $u_n < \frac{u_m}{r^m}.r^n$ for all $n > m$.

$\frac{u_m}{r^m}$ is a positive number and Σr^n is a geometric series of common ratio r where $0 < r < 1$ and therefore Σr^n is convergent. Therefore Σu_n is convergent by Comparison test.

Case 2. $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that

$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon$ for all $n \geq k$.

Let $l - \epsilon = p$. Then $p > 1$.

We have $\frac{u_{k+1}}{u_k} > p, \frac{u_{k+2}}{u_{k+1}} > p, \cdots, \frac{u_n}{u_{n-1}} > p$, where $n > k$.

Consequently, $\frac{u_n}{u_k} > p^{n-k}$ for all $n > k$ or, $u_n > \frac{u_k}{p^k}.p^n$ for all $n > k$.

$\frac{u_k}{p^k}$ is a positive number and Σp^n is a geometric series of common ratio $p > 1$ and therefore Σp^n is divergent. Therefore Σu_n is divergent by Comparison test.

This completes the proof.

Note. When $l = 1$, the test fails to give a decision.

Let $u_n = \frac{1}{n}$. Then Σu_n is a divergent series and $\lim \frac{u_{n+1}}{u_n} = 1$.

Let $u_n = \frac{1}{n^2}$. Then Σu_n is a convergent series and $\lim \frac{u_{n+1}}{u_n} = 1$.

Although for both the series $\lim \frac{u_{n+1}}{u_n} = 1$, one is a convergent series and the other is a divergent series.

Therefore if $\lim \frac{u_{n+1}}{u_n} = 1$, nothing can be said about the convergence or divergence of the series Σu_n.

Theorem 6.3.5. Cauchy's root test.

Let Σu_n be a series of positive real numbers and let $\lim u_n^{1/n} = l$.

Then Σu_n is convergent if $l < 1$, Σu_n is divergent if $l > 1$.

Proof. **Case 1.** $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim u_n^{1/n} = l$, there exists a natural number m such that

$l - \epsilon < u_n^{1/n} < l + \epsilon$ for all $n \geq m$

or, $(l - \epsilon)^n < u_n < (l + \epsilon)^n$ for all $n \geq m$.

Let $l + \epsilon = r$. Then $o < r < 1$ and $u_n < r^n$ for all $n \geq m$.

But Σr^n is a geometric series of common ratio r where $0 < r < 1$. So Σr^n is convergent.

Therefore Σu_n is convergent by Comparison test.

Case 2. $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim u_n^{1/n} = l$, there exists a natural number k such that

$l - \epsilon < u_n^{1/n} < l + \epsilon$ for all $n \geq k$

or, $(l - \epsilon)^n < u_n < (l + \epsilon)^n$ for all $n \geq k$.

Let $l - \epsilon = p$. Then $p > 1$ and $u_n > p^n$ for all $n \geq k$.

But Σp^n is a geometric series of common ratio $p > 1$. So Σp^n is divergent.

Therefore Σu_n is divergent by Comparison test.

This completes the proof.

Note. When $l = 1$, the test fails to give a decision.

Let $u_n = 1/n$. Then $\lim u_n^{1/n} = 1$ and Σu_n is a divergent series.

Let $u_n = 1/n^2$. Then $\lim u_n^{1/n} = 1$ and Σu_n is a convergent series.

Although for both the series $\lim u_n^{1/n} = 1$, one is a convergent series and the other is a divergent series.

Thus if $\lim u_n^{1/n} = 1$, nothing can be said about the convergence or divergence of the series Σu_n.

Worked Examples (continued).

4. Test the convergence of the series $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{2n-1}{n!}$.

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)(2n-1)}$ and $\lim \frac{u_{n+1}}{u_n} = 0 < 1$.

By D'Alembert's ratio test, Σu_n is convergent.

5. Examine the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \cdots, x > 0$.

Let $\sum_{n=1}^{\infty} u_n$ be the given series. Since $x > 0$, Σu_n is a series of positive terms. $\frac{u_{n+1}}{u_n} = \frac{nx}{n+1}$ and $\lim \frac{u_{n+1}}{u_n} = x$.

By D'Alembert's ratio test, Σu_n is convergent if $x < 1$ and Σu_n is divergent if $x > 1$.

When $x = 1$, the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots$ and this is divergent.

6. Test the convergence of the series $1 + \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \cdots \cdots$

Ignoring the first term, let $\sum_{n=1}^{\infty} u_n$ be the series. Then $u_n = \frac{n^n}{n!}$.

$\frac{u_{n+1}}{u_n} = (\frac{n+1}{n})^n$ and $\lim \frac{u_{n+1}}{u_n} = e > 1$.

Σu_n is divergent by D'Alembert's ratio test.

Therefore the given series is divergent.

7. Test the convergence of the series

$1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \cdots \cdots$

Here $u_n = \{2^{n+(-1)^n}\}^{-1}$ and $\lim u_n^{1/n} = \lim\{2^{1+\frac{(-1)^n}{n}}\}^{-1} = \frac{1}{2}$.

Therefore the series is convergent by Cauchy's root test.

Note. Here $\frac{u_{n+1}}{u_n} = \frac{1}{8}$ if n be odd,

$= 2$ if n be even.

$\lim \frac{u_{n+1}}{u_n}$ does not exist and therefore the converge nce of the series cannot be decided by D'Alembert's ratio test.

It follows that the root test is more powerful tha the ratio test in deciding convergence of a series of positive real numbers.

The fact is explained by the theorem 5.16.4 which states that if $u_n > 0$ then

$\underline{\lim} \frac{u_{n+1}}{u_n} \leq \underline{\lim}\, u_n^{1/n} \leq \overline{\lim}\, u_n^{1/n} \leq \overline{\lim} \frac{u_{n+1}}{u_n}$.

If for some series Σu_n of positive terms $\lim \frac{u_{n+1}}{u_n}$ exists and equals l, then $\lim u_n^{1/n}$ also exists and equals l. Therefore when the ratio test decides the convergence of the series Σu_n, the root test also does.

But if for some series Σu_n of positive terms $\lim u_n^{1/n}$ exists and equals l, then $\lim \frac{u_{n+1}}{u_n}$ does not necessarily exist. Therefore when the root test decides the convergence of the series Σu_n, the ratio test may fail to do so.

Theorem 6.3.6. General form of ratio test.

Let Σu_n be a series of positive real numbers and let

$\overline{\lim}\, \frac{u_{n+1}}{u_n} = R$, $\underline{\lim}\, \frac{u_{n+1}}{u_n} = r$.

Then Σu_n is convergent if $R < 1$, Σu_n is divergent if $r > 1$.

Proof. **Case 1.** $R < 1$.

Let us choose a positive ϵ such that $R + \epsilon < 1$.

Since $\overline{\lim}\, \frac{u_{n+1}}{u_n} = R$, there exists a natural number m such that

$\frac{u_{n+1}}{u_n} < R + \epsilon$ for all $n \geq m$.

Let $R + \epsilon = p$. Then $0 < p < 1$.

We have $\frac{u_{m+1}}{u_m} < p, \frac{u_{m+2}}{u_{m+1}} < p, \cdots, \frac{u_n}{u_{n-1}} < p$, where $n > m$.

Consequently, $\frac{u_n}{u_m} < p^{n-m}$ for all $n > m$

or, $u_n < \frac{u_m}{p^m} p^n$ for all $n > m$.

$\frac{u_m}{p^m}$ is a positive number and Σp^n is convergent, since $0 < p < 1$.

Therefore Σu_n is convergent by Comparison test.

Case 2. $r > 1$.

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\underline{\lim}\, \frac{u_{n+1}}{u_n} = r$, there exists a natural number k such that

$\frac{u_{n+1}}{u_n} > r - \epsilon$ for all $n \geq k$.

Let $r - \epsilon = q$. Then $q > 1$.

We have $\frac{u_{k+1}}{u_k} > q, \frac{u_{k+2}}{u_{k+1}} > q, \cdots, \frac{u_n}{u_{n-1}} > q$, where $n > k$.

Consequently, $\frac{u_n}{u_k} > q^{n-k}$ for all $n > k$

or, $u_n > \frac{u_k}{q^k} q^n$ for all $n > k$.

$\frac{u_k}{q^k}$ is a positive number and Σq^n is divergent, since $q > 1$.Therefore Σu_n is divergent by Comparison test.

This completes the proof.

Theorem 6.3.7. General form of root test.

Let Σu_n be a series of positive real numbers and let $\overline{\lim}\, u_n^{1/n} = r$.

Then Σu_n is convergent if $r < 1$, Σu_n is divergent if $r > 1$.

Proof. **Case 1.** $r < 1$.

Let us choose a positive ϵ such that $r + \epsilon < 1$.

Since $\overline{\lim}\, u_n^{1/n} = r$, there exists a natural number m such that

$u_n^{1/n} < r + \epsilon$ for all $n \geq m$.

Let $r + \epsilon = p$. Then $0 < p < 1$ and $u_n < p^n$ for all $n \geq m$.

But Σp^n is a convergent series since $0 < p < 1$.

Therefore Σu_n is convergent by Comparison test.

Case 2. $r > 1$.

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\overline{\lim}\, u_n^{1/n} = r$, $u_n^{1/n} > r - \epsilon$ for infinitely many n.

That is, infinite number of elements of the sequence (u_n) are greater than 1 and therefore $\lim u_n$ cannot be 0.

Therefore Σu_n is divergent, since a necessary condition for convergence of the series Σu_n is $\lim u_n = 0$.

This completes the proof.

Worked Examples (continued).

8. Test the convergence of the series.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\cdots$

Let $\sum_{1}^{\infty} u_n$ be the given series.

Then $u_{2n} = \frac{1}{3^n}$, $u_{2n+1} = \frac{1}{2^{n+1}}$, $u_{2n-1} = \frac{1}{2^n}$.

$\lim \frac{u_{2n}}{u_{2n-1}} = \lim(\frac{2}{3})^n = 0$, $\lim \frac{u_{2n+1}}{u_{2n}} = \lim \frac{1}{3}(\frac{3}{2})^{n+1} = \infty$.

It follows that $\limsup \frac{u_{n+1}}{u_n} = \infty$, $\liminf \frac{u_{n+1}}{u_n} = 0$.

Clearly, the ratio test gives no decision.

$\lim(u_{2n})^{1/2n} = \frac{1}{\sqrt{3}}$, $\lim(u_{2n+1})^{1/(2n+1)} = \frac{1}{\sqrt{2}}$.

It follows that $\limsup(u_n)^{1/n} = \frac{1}{\sqrt{2}} < 1$.

Therefore Σu_n is convergent by the root test.

9. Test the series

$a + b + a^2 + b^2 + a^3 + b^3 + \cdots\cdots$ where $0 < a < b < 1$.

Let $\sum_{1}^{\infty} u_n$ be the given series.

Here $\frac{u_{2n}}{u_{2n-1}} = (\frac{b}{a})^n$, $\frac{u_{2n+1}}{u_{2n}} = a(\frac{a}{b})^n$. $\lim \frac{u_{2n}}{u_{2n-1}} = \infty$, $\lim \frac{u_{2n+1}}{u_{2n}} = 0$.

It follows that $\overline{\lim}\, \frac{u_{n+1}}{u_n} = \infty$, $\underline{\lim}\, \frac{u_{n+1}}{u_n} = 0$.

The ratio test gives no decision.

$\lim u_{2n}^{1/2n} = \lim(b^n)^{1/2n} = \sqrt{b}$,

$\lim(u_{2n+1})^{1/(2n+1)} = \lim(a^{n+1})^{\frac{1}{2n+1}} = \sqrt{a}$.

It follows that $\limsup(u_n)^{1/n} = \sqrt{b} < 1$.

Therefore Σu_n is convergent by the root test.

Note. Here the ratio test does not decide convergence of the series but the root test does. The root test is more powerful than the ratio test for deciding convergence of a series positive real numbers.

Theorem 6.3.8. Cauchy's condensation test.

Let $(f(n))$ be a monotone decreasing sequence of positive real numbers and a be a positive integer > 1.

Then the series $\sum_1^\infty f(n)$ and $\sum_1^\infty a^n f(a^n)$ converge or diverge together.

Proof. Grouping the terms of $\Sigma f(n)$ as

$\{f(1)\}+\{f(2)+\cdots+f(a)\}+\{f(a+1)+\cdots+f(a^2)\}+\cdots$ and ignoring the first term, let Σv_n be the new series.

Then $v_n = f(a^{n-1}+1) + f(a^{n-1}+2) + \cdots + f(a^n)$ for all $n \geq 1$.

The number of terms in v_n is $a^n - a^{n-1}$. Since $(f(n))$ is a monotone decreasing sequence, each term of $v_n \leq f(a^{n-1}+1)$ and $\geq f(a^n)$.

Therefore $(a^n - a^{n-1})f(a^n) \leq v_n$ for all $n \geq 1$

or, $\frac{a-1}{a} a^n f(a^n) \leq v_n$ for all $n \geq 1$.

Let $w_n = a^n f(a^n)$. Then $w_n \leq \frac{a}{a-1} v_n$ for all $n \geq 1$.

$\frac{a}{a-1}$ is positive. By Comparison test, Σw_n is convergent if Σv_n is convergent and Σv_n is divergent if Σw_n is divergent. ... ... (A)

$$\begin{aligned} \text{Again, } v_n &\leq (a^n - a^{n-1})f(a^{n-1}+1) \\ &\leq (a^n - a^{n-1})f(a^{n-1}) \text{ for all } n \geq 2. \end{aligned}$$

That is, $v_n \leq (a-1)w_{n-1}$ for all $n \geq 2$.

$a-1$ is positive. By Comparison test, Σv_n is convergent if Σw_n is convergent and Σw_n is divergent if Σv_n is divergent. ... ... (B)

From (A) and (B), Σv_n and Σw_n converge or diverge together.

But Σv_n and $\Sigma f(n)$ converge or diverge together.

Therefore $\Sigma f(n)$ and Σw_n, i.e., $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ converge or diverge together.

This completes the proof.

Worked Examples (continued).

10. Test the convergence of the series $\sum_1^\infty \frac{1}{n}$.

Let $f(n) = \frac{1}{n}$. Then $(f(n))$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$2^n f(2^n) = 1$ and therefore $\Sigma 2^n f(2^n)$ is divergent.

It follows that $\Sigma f(n)$ is divergent, i.e., $\Sigma \frac{1}{n}$ is divergent.

11. Discuss the convergence of the series $\sum_1^\infty \frac{1}{n^p}$, $p > 0$.

Let $f(n) = \frac{1}{n^p}$. As $p > 0$, the sequence $(f(n))$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$2^n f(2^n) = 2^n . \frac{1}{2^{np}} = \frac{1}{2^{n(p-1)}}$.

$\Sigma (\frac{1}{2^{p-1}})^n$ is a geometric series and it converges if $p > 1$ and diverges if $p \leq 1$. Therefore $\sum_1^\infty \frac{1}{n^p}$ is convergent when $p > 1$ and is divergent when $0 < p \leq 1$.

12. Discuss the convergence of the series $\sum_2^\infty \frac{1}{n(\log n)^p}$, $p > 0$.

Let $f(n) = \frac{1}{n(\log n)^p}, n \geq 2$. As $(\log n)$ is an increasing sequence and $p > 0$, $\{\log(n+1)\}^p > \{\log n\}^p$ and therefore $(n+1)\{\log(n+1)\}^p > n(\log n)^p$.

Therefore $(f(n))_2^\infty$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$\Sigma 2^n f(2^n) = \Sigma \frac{1}{(n \log 2)^p}$ and this converges when $p > 1$ and diverges when $p \leq 1$. Therefore $\sum_{n=2}^\infty f(n)$ is convergent when $p > 1$ and divergent when $0 < p \leq 1$.

If the limits $\lim \frac{u_{n+1}}{u_n}$ or $\lim \sqrt[n]{u_n}$ be equal to 1, D'Alembert's ratio test and Cauchy's root test fail to decide convergence of the series Σu_n. In such cases it is often helpful to use a more delicate test due to Raabe.

Theorem 6.3.9. Raabe's test.

Let Σu_n be a series of positive real numbers and $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$. Then Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

Proof. **Case 1.** $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$, there exists a natural number m such that

$$l - \epsilon < n(\frac{u_n}{u_{n+1}} - 1) < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = r$. Then $r > 1$.

We have $nu_n - nu_{n+1} > ru_{n+1}$ for all $n \geq m$

or, $nu_n - (n+1)u_{n+1} > (r-1)u_{n+1}$ for all $n \geq m$.

We have $mu_m - (m+1)u_{m+1} > (r-1)u_{m+1}$
$(m+1)u_{m+1} - (m+2)u_{m+2} > (r-1)u_{m+2}$
$\cdots \quad \cdots$
$(n-1)u_{n-1} - nu_n > (r-1)u_n$, where $n > m$.
Consequently, $mu_m - nu_n > (r-1)(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

$$\text{or, } u_{m+1} + u_{m+2} + \cdots + u_n < \frac{1}{r-1}(mu_m - nu_n) < \frac{1}{r-1}mu_m$$

or, $s_n - s_m < \frac{1}{r-1}mu_m$, where $s_n = u_1 + u_2 + \cdots + u_n$
or, $s_n < \frac{1}{r-1}mu_m + s_m$ for all $n > m$.
This shows that the sequence (s_n) is bounded above and therefore the series Σu_n is convergent.

Case 2. $l < 1$.
Let us choose a positive ϵ such that $l+\epsilon < 1$. Since $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$, there exists a natural number k such that

$$l - \epsilon < n(\frac{u_n}{u_{n+1}} - 1) < l + \epsilon \text{ for all } n \geq k.$$

Let $l + \epsilon = p$. Then $p < 1$.
We have $n(\frac{u_n}{u_{n+1}} - 1) < p < 1$ for all $n \geq k$.
Therefore $n(u_n - u_{n+1}) < pu_{n+1} < u_{n+1}$ for all $n \geq k$
or, $nu_n < (n+1)u_{n+1}$ for all $n \geq k$.
We have $ku_k < (k+1)u_{k+1}$
$(k+1)u_{k+1} < (k+2)u_{k+2}$
$\cdots \quad \cdots$
$(n-1)u_{n-1} < nu_n$, where $n > k$.
Consequently, $nu_n > ku_k$ for all $n > k$
or, $u_n > ku_k.\frac{1}{n}$.
ku_k is a positive number and $\Sigma\frac{1}{n}$ is a divergent series. Therefore Σu_n is divergent by Comparison test.
This completes the proof.

Note. If $l = 1$, the test is inconclusive. This can be established by taking the series Σu_n, where $u_n > 0$ for all $n \in \mathbb{N}$ and $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{2}{n \log n}, n \geq 2$; and the series Σv_n, where $v_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.

Theorem 6.3.10. General form of Raabe's test.
Let Σu_n be a series of positive real numbers and let $\overline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = R$ and $\underline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = r$.
Then Σu_n is convergent if $r > 1$, Σu_n is divergent if $R < 1$.

Proof. **Case 1.** $r > 1$.
Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\underline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = r$, there exists a natural number m such that $n(\frac{u_n}{u_{n+1}} - 1) > r - \epsilon$ for all $n \geq m$.

Let $r - \epsilon = k$. Then $k > 1$.
We have $nu_n - nu_{n+1} > ku_{n+1}$ for all $n \geq m$
or, $nu_n - (n+1)u_{n+1} > (k-1)u_{n+1}$ for all $n \geq m$
We have $mu_m - (m+1)u_{m+1} > (k-1)u_{m+1}$
$(m+1)u_{m+1} - (m+2)u_{m+2} > (k-1)u_{m+2}$
$\cdots \quad \cdots$
$(n-1)u_{n-1} - nu_n > (k-1)u_n$, where $n > m$.

Consequently, $mu_m - nu_n > (k-1)(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

$$\text{or, } u_{m+1} + u_{m+2} + \cdots + u_n < \frac{1}{k-1}(mu_m - nu_n) < \frac{1}{k-1}mu_m.$$

Let $s_n = u_1 + u_2 + \cdots + u_n$.
Then $s_n < s_m + \frac{1}{k-1}mu_m$ for all $n > m$.

This shows that the sequence (s_n) is bounded above and therefore the series Σu_n is convergent.

Case 2. $R < 1$.
Let us choose a positive ϵ such that $R + \epsilon < 1$.
Since $\overline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = R$, there exists a natural number k such that $n(\frac{u_n}{u_{n+1}} - 1) < R + \epsilon$ for all $n \geq k$.

Let $R + \epsilon = p$. Then $p < 1$.
We have $n(u_n - u_{n+1}) < pu_{n+1}$ for all $n \geq k$
i.e., $n(u_n - u_{n+1}) < u_{n+1}$ for all $n > k$
or, $nu_n < (n+1)u_{n+1}$ for all $n \geq k$.

We have $ku_k < (k+1)u_{k+1}$
$(k+1)u_{k+1} < (k+2)u_{k+2}$
$\cdots \quad \cdots$
$(n-1)u_{n-1} < nu_n$, where $n > k$.
Therefore $ku_k < nu_n$ for all $n > k$
or, $u_n > ku_k.\frac{1}{n}$ for all $n > k$.
ku_k is positive and $\Sigma 1/n$ is a divergent series. Therefore Σu_n is divergent by Comparison test.

This completes the proof.

Worked Example (continued).
13. Test the convergence of the series
$1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)} \cdot \frac{1}{2n-1}$ for all $n \geq 2$. $\frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(2n+1)}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.

D'Alembert's ratio test gives no decision.

Let us apply Raabe's test.

$\lim\limits_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) = \lim\limits_{n\to\infty} \frac{6n^2-n}{(2n-1)^2} = \frac{3}{2} > 1$.

By Raabe's test, Σu_n is convergent.

Theorem 6.3.11. Logarithmic test.

Let Σu_n be a series of positive real numbers and $\lim n \log \frac{u_n}{u_{n+1}} = l$. Then Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

Proof. **Case 1.** $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number m such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = r$. Then $r > 1$.

We have $n \log \frac{u_n}{u_{n+1}} > r > 1$ for all $n \geq m$

or, $\frac{u_n}{u_{n+1}} > e^{r/n}$ for all $n \geq m$.

Since $((1+\frac{1}{n})^n)$ is a monotonic increasing sequence converging to e and e is irrational, $(1+\frac{1}{n})^n < e$ for all $n \in \mathbb{N}$.

It follows that $\frac{u_n}{u_{n+1}} > (1+\frac{1}{n})^r$ for all $n \geq m$

$= \frac{(n+1)^r}{n^r}$.

Let $v_n = \frac{1}{n^r}$. Then $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ for all $n \geq m$.

That is, $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ for all $n \geq m$.

By Comparison test, Σu_n is convergent since Σv_n is convergent.

Case 2. $0 \leq l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number p such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq p.$$

Let $l + \epsilon = k$. Then $0 < k < 1$.

$n \log \frac{u_n}{u_{n+1}} < k$ for all $n \geq p$

or, $\frac{u_n}{u_{n+1}} < e^{k/n}$ for all $n \geq p$.

Since $((1+\frac{1}{n-1})^n)_{n=2}^{\infty}$ is a monotone decreasing sequence converging to e and e is irrational, $(1+\frac{1}{n-1})^n > e$ for all $n \geq 2$.

Therefore $\frac{u_n}{u_{n+1}} < (\frac{n}{n-1})^k$ for all $n \geq p > 1$.

Let $w_n = \frac{1}{(n-1)^k}$ for $n \geq 2$. Then $\sum\limits_{2}^{\infty} w_n$ is divergent and $\frac{u_{n+1}}{u_n} > \frac{w_{n+1}}{w_n}$ for all $n \geq p > 1$.

By Comparison test, Σu_n is divergent.

Case 3. $l < 0$.

Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number q such that

$$n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq q.$$

Let $l + \epsilon = s$. Then $s < 0$ and $n \log \frac{u_n}{u_{n+1}} < s < 0$ for all $n \geq q$

or, $n \log \frac{u_{n+1}}{u_n} > p' > 0$ (where $p' = -s$) for all $n \geq q$

or, $\frac{u_{n+1}}{u_n} > e^{p'/n}$ for all $n \geq q$.

Since $e > (1+\frac{1}{n})^n$ for all $n \in \mathbb{N}$, it follows that $\frac{u_{n+1}}{u_n} > (1+\frac{1}{n})^{p'}$ for all $n \geq q$.

Let $w_n = n^{p'}$. Then $\frac{u_{n+1}}{u_n} > \frac{w_{n+1}}{w_n}$ for all $n \geq q$.

As Σw_n is a divergent series, Σu_n is divergent by Compasiron test.

This completes the proof.

Worked Example (continued).

14. Test the convergence of the series

$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots \quad \cdots, x > 0$

Ignoring the first term, let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{n^n x^n}{n!}$. $\frac{u_{n+1}}{u_n} = (1+\frac{1}{n})^n x$ and $\lim \frac{u_{n+1}}{u_n} = ex$.

By D'Alembert's ratio test,

Σu_n is convergent if $0 < x < 1/e$, Σu_n is divergent if $x > 1/e$.

When $x = 1/e$, let us apply Logarithmic test.

$\lim n \log \frac{u_n}{u_{n+1}} = \lim n[1 + n \log \frac{n}{n+1}] = \lim[n + n^2 \log \frac{n}{n+1}] = \frac{1}{2}$.

$[\lim\limits_{x\to\infty} [x + x^2 \log \frac{x}{x+1}] = \frac{1}{2} \Rightarrow \lim\limits_{n\to\infty} [n + n^2 \log \frac{n}{n+1}] = \frac{1}{2}$, by sequential criterion for limits.]

By Logarithmic test, Σu_n is divergent when $x = 1/e$.

So the given series is convergent if $0 < x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

Theorem 6.3.12. Kummer's test.

Let Σu_n and $\Sigma 1/d_n$ be two series of positive real numbers and let $w_n = \frac{u_n}{u_{n+1}} d_n - d_{n+1}$.

If $\lim w_n = k > 0$, then Σu_n is convergent.

If $\lim w_n = k < 0$ and $\Sigma 1/d_n$ is divergent, then Σu_n is divergent.

Proof. **Case 1.** $k > 0$.

Let us choose a positive ϵ such that $k - \epsilon > 0$.

Since $\lim w_n = k$, there exists a natural number m such that

$$k - \epsilon < w_n < k + \epsilon \text{ for all } n \geq m.$$

Let $k - \epsilon = r$. Then $r > 0$ and $\frac{u_n d_n}{u_{n+1}} - d_{n+1} > r$ for all $n \geq m$

or, $u_n d_n - u_{n+1} d_{n+1} > r u_{n+1}$ for all $n \geq m$.

Then we have $u_m d_m - u_{m+1} d_{m+1} > r u_{m+1}$

$u_{m+1} d_{m+1} - u_{m+2} d_{m+2} > r u_{m+2}$

$\cdots \quad \cdots \quad \cdots$

$u_{n-1} d_{n-1} - u_n d_n > r u_n$, where $n > m$.

So $u_m d_m - u_n d_n > r(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

or, $u_{m+1} + u_{m+2} + \cdots + u_n < \frac{1}{r}(u_m d_m - u_n d_n)$

$< \frac{1}{r} u_m d_m$.

or, $s_n - s_m < \frac{1}{r} u_m d_m$, where $s_n = u_1 + u_2 + \cdots + u_n$

or, $s_n < s_m + \frac{1}{r} u_m d_m$ for all $n > m$.

The sequence (s_n) is bounded above and therefore Σu_n is convergent.

Case 2. $k < 0$.

Let us choose a positive ϵ such that $k + \epsilon < 0$.

Then there exists a natural number p such that

$$k - \epsilon < w_n < k + \epsilon \text{ for all } n \geq p.$$

This gives $\frac{u_n d_n}{u_{n+1}} - d_{n+1} < 0$ for all $n \geq p$

or, $u_n d_n < u_{n+1} d_{n+1}$ for all $n \geq p$.

We have $u_p d_p < u_{p+1} d_{p+1}$

$u_{p+1} d_{p+1} < u_{p+2} d_{p+2}$

$\cdots \quad \cdots \quad \cdots$

$u_{n-1} d_{n-1} < u_n d_n$ for all $n > p$.

So $u_p d_p < u_n d_n$ for all $n > p$

or, $u_n > \frac{u_p d_p}{d_n}$ for all $n > p$.

$u_p d_p$ is positive and $\Sigma \frac{1}{d_n}$ is a divergent series.

Therefore Σu_n is divergent by Comparison test.

Corollary 1. If we take $d_n = n$, then

$w_n = n\frac{u_n}{u_{n+1}} - (n+1) = n(\frac{u_n}{u_{n+1}} - 1) - 1$.

$\lim w_n = \lim n(\frac{u_n}{u_{n+1}} - 1) - 1$.

Let $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$. Then Kummer's test gives

Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

This is Rabbe's test.

Corollary 2. If we take $d_n = 1$, then $w_n = \frac{u_n}{u_{n+1}} - 1$ and

$\lim w_n = \lim(\frac{u_n}{u_{n+1}} - 1)$.

Let $\lim \frac{u_{n+1}}{u_n} = l$. Then Kummer's test gives

Σu_n is convergent if $\frac{1}{l} > 1$, i.e., if $l < 1$; Σu_n is divergent if $\frac{1}{l} < 1$, i.e., if $l > 1$.

This is D'Alembert's ratio test.

Theorem 6.3.13. Gauss's test.

Let Σu_n be a series of positive real numbers and let $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p}$, where $p > 1$ and the sequence (b_n) is bounded.

Then Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \leq 1$.

Proof. **Case 1.** $a \neq 1$.

$\lim n(\frac{u_n}{u_{n+1}} - 1) = \lim n(\frac{a}{n} + \frac{b_n}{n^p}) = a$, since $\lim \frac{b_n}{n^{p-1}} = 0$.

By Raabe's test, Σu_n is convergent if $a > 1$ and Σu_n is divergent if $a < 1$.

Case 2. $a = 1$.

Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n^p}$.

Let us apply Kummer's test by taking $d_n = n \log n$.

$$\begin{aligned} \text{Then } w_n &= \frac{d_n u_n}{u_{n+1}} - d_{n+1} \\ &= n \log n(1 + \tfrac{1}{n} + \tfrac{b_n}{n^p}) - (n+1)\log(n+1) \\ &= (n+1)\log n + \tfrac{b_n \log n}{n^{p-1}} - (n+1)\log(n+1) \\ &= (n+1)\log \tfrac{n}{n+1} + \tfrac{\log n}{n^{p-1}} b_n. \end{aligned}$$

$$\begin{aligned} \lim w_n &= \lim(n+1)\log(1 - \tfrac{1}{n+1}) + \lim \tfrac{\log n}{n^{p-1}}.b_n \\ &= -1, \text{ since } \lim \log(1 - \tfrac{1}{n+1})^{n+1} = \log e^{-1} = -1 \text{ and} \end{aligned}$$

$\lim \frac{\log n}{n^{p-1}} = 0$ and (b_n) is a bounded sequence.

By Kummer's test, Σu_n is divergent.

Worked Examples (continued).

15. Examine the convergence of the series $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{2^2.4^2 \cdots (2n-2)^2}{3^2.5^2 \cdots (2n-1)^2}$ for all $n \geq 2$.

$\frac{u_{n+1}}{u_n} = \frac{4n^2}{4n^2+4n+1}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.

D'Alembert's ratio test gives no decision.

Let us apply Raabe's test.

$\lim n(\frac{u_n}{u_{n+1}} - 1) = \lim n(\frac{4n+1}{4n^2+4n+1}) = 1$.

Raabe's test gives no decision.

Let us apply Gauss's test.

$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{4n^2}$.

$\frac{u_n}{u_{n+1}}$ is of the form $1 + \frac{a}{n} + \frac{b_n}{n^2}$, where $a = 1$ and $b_n = \frac{1}{4}$ and so (b_n) is a bounded sequence.

By Gauss's test, Σu_n is divergent.

16. Hypergeometric series.

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots,$$

where $\alpha, \beta, \gamma, x > 0$.

Ignoring the first term, let $\sum\limits_{1}^{\infty} u_n$ be the series.

Then $u_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1.2\cdots.n\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n$ for $n \geq 1$.

$\frac{u_{n+1}}{u_n} = \frac{(\alpha+n)(\beta+n)}{(1+n)(\gamma+n)}x$ and $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = x$.

By D'Alembert's ratio test, Σu_n is convergent if $0 < x < 1$ and Σu_n is divergent if $x > 1$.

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}$

$$= 1 + \left(\frac{(\gamma+1-\alpha-\beta)n+(\gamma-\alpha\beta)}{n^2+(\alpha+\beta)n+\alpha\beta}\right)$$

$$= 1 + \left(\frac{\gamma+1-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta}{n^2}\right)\left[1 + \frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right]^{-1}$$

$$= 1 + \left(\frac{\gamma+1-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta}{n^2}\right)\left[1 - \frac{\alpha+\beta}{n} - \frac{\alpha\beta}{n^2} + \cdots\right]$$

$= 1+\frac{\gamma+1-\alpha-\beta}{n}+\frac{1}{n^2}[(\gamma-\alpha\beta)-(\alpha+\beta)(\gamma+1-\alpha-\beta)+$ terms containing $\frac{1}{n}$ and higher powers of $\frac{1}{n}]$

$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + \frac{\phi(n)}{n^2}$, where $\lim\limits_{n\to\infty} \phi(n)$ is finite and therefore $(\phi(n))$ is bounded.

By Gauss's test, when $x = 1$, Σu_n is convergent if $\gamma + 1 - \alpha - \beta > 1$ and Σu_n is divergent if $\gamma + 1 - \alpha - \beta \leq 1$.

Therefore the series is convergent if $0 < x < 1$ and divergent if $x > 1$.

When $x = 1$, the series is convergent if $\gamma > \alpha + \beta$ and divergent if $\gamma \leq \alpha + \beta$.

The order symbol O.

Let f and ϕ be two functions of n defined for all $n \geq m$, m being a natural number; and ϕ be ultimately monotone with $\phi(n) > 0$ for sufficiently large n.

If there exists a natural number $m_o \geq m$ such that $|f(n)| \leq k\phi(n)$ for all $n \geq m_o$, k being a positive constant, we write $f = O(\phi)$.

Thus $O(\phi)$ denotes a function f such that $f(n) = h(n)\phi(n)$ where h is a bounded function of n.

In particular, $f = O(1)$ means that f is a bounded function of n.

Examples.

1. Let $f(n) = 5n^2 + 3n + 1$. Then $f(n) = O(n^2)$, since $f(n) \leq 5n^2$ for all $n \geq 1$.

2. Let $f(n) = \frac{(-1)^n}{n}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{|f(n)|}{\frac{1}{n}} \to 1$ as $n \to \infty$

3. Let $f(n) = \frac{2n^2-3n+1}{5n^3-3}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{f(n)}{\frac{1}{n}} \to \frac{2}{5}$ as $n \to \infty$.

4. Let $f(n) = \frac{1}{\sqrt{n^2-1}+\sqrt{n^2+1}}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{f(n)}{\frac{1}{n}} \to \frac{1}{2}$ as $n \to \infty$.

5. Let $f(n) = 2\sin\frac{n\pi}{4}$. Then $f(n) = O(1)$, since $|f(n)| \leq 2$ for all $n \geq 1$.

Alternative form of Gauss's test.

Let Σu_n be a series of positive real numbers and let $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + O(\frac{1}{n^p})$ where $p > 1$.

Then Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \leq 1$.

$O(\frac{1}{n^p})$ denotes a sequence $(f(n))$ such that $f(n) = h(n).\frac{1}{n^p}$, where $(h(n))$ is a bounded sequence.

Therefore $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{h(n)}{n^p}$, where $(h(n))$ is a bounded sequence and $p > 1$.

By Gauss's test, Σu_n is convergent if $a > 1$. Σu_n is divergent if $a \geq 1$.

Worked Example (continued).

17. Test the convergence of the series

$(\frac{1}{2})^2 + (\frac{1.3}{2.4})^2 + (\frac{1.3.5}{2.4.6.})^2 + \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \{\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\}^2$ for all $n \geq 1$.

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = (1 + \tfrac{1}{n})^2(1 + \tfrac{1}{2n})^{-2}$$
$$= (1 + \tfrac{2}{n} + \tfrac{1}{n^2})[1 - \tfrac{1}{n} + O(\tfrac{1}{n^2})]$$
$$= 1 + \tfrac{1}{n} + O(\tfrac{1}{n^2}).$$

By Gauss's test, Σu_n is divergent.

Theorem 6.3.14. De Morgan and Bertrand's test.

Let Σu_n be a series of positive real numbers and $\lim[n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n = l$.

Then Σu_n is convergent if $l > 1$; and divergent if $l < 1$.

Proof. Let $b_n = [n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n$.

Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n \log n}$.

Let $w_n = \frac{u_n}{u_{n+1}} d_n - d_{n+1}$ where $d_n = n \log n$.

Then $\sum_{n=2}^{\infty} \frac{1}{d_n}$ is a divergent series and

$$\begin{aligned} w_n &= \frac{u_n}{u_{n+1}} n \log n - (n+1)\log(n+1) \\ &= (1 + \frac{1}{n} + \frac{b_n}{n \log n}) n \log n - (n+1)\log(n+1) \\ &= (n+1)\log\frac{n}{n+1} + b_n. \end{aligned}$$

$\lim w_n = \lim \log \frac{1}{(1+\frac{1}{n})^{n+1}} + l = -1 + l$.

By Kummer's test, Σu_n is convergent if $l - 1 > 0$, i.e., if $l > 1$ and Σu_n is divergent if $l - 1 < 0$, i.e., if $l < 1$.

Worked Example (continued).

18. Test the convergence of the series

$(\frac{1}{2})^3 + (\frac{1.4}{2.5})^3 + (\frac{1.4.7}{2.5.8})^3 + \cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \{\frac{1.4.7...(3n-2)}{2.5.8...(3n-1)}\}^3$.

$\frac{u_{n+1}}{u_n} = (\frac{3n+1}{3n+2})^3$.

$$\begin{aligned} \lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) &= \lim n[\frac{(3n+2)^3-(3n+1)^3}{(3n+1)^3}] \\ &= \lim \frac{27n^3+27n^2+7n}{27n^3+27n^2+9n+1} = 1. \end{aligned}$$

Raabe's test gives no decision.

Let us apply De Morgan and Bertrand's test.

$$\begin{aligned} \lim_{n\to\infty} [n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n &= \lim \frac{(-2n-1)\log n}{27n^3+27n^2+9n+1} \\ &= \lim \frac{-2n^3-n^2}{27n^3+27n^2+9n+1} \cdot \frac{\log n}{n^2} \\ &= \frac{-2}{27}.0, \text{ since } \lim \frac{\log n}{n^2} = 0 \\ &= 0 < 1. \end{aligned}$$

By De Morgan and Bertrand's test, Σu_n is divergent.

Theorem 6.3.15. Abel's theorem or Pringsheim's theorem.

If Σu_n be a convergent series positive real numbers and (u_n) is a monotone decreasing sequence, then $\lim nu_n = 0$.

Proof. Since Σu_n is convergent, for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \frac{\epsilon}{2}$ for all $n \geq m$ and for every natural number p.

Let $n = m$.

Then $u_{m+1} + u_{m+2} + \cdots + u_{m+p} < \frac{\epsilon}{2}$ for every natural number p.

But $u_{m+1} + u_{m+2} + \cdots + u_{m+p} \geq pu_{m+p}$, since (u_n) is a monotone decreasing sequence.

Consequently, $pu_{m+p} < \frac{\epsilon}{2}$ for every natural number p.

Let $p = m$. Then $2mu_{2m} < \epsilon$ ... ... (i)

Let $p = m + 1$. Then $(m+1)u_{2m+1} < \frac{\epsilon}{2}$.

Therefore $(2m+1)u_{2m+1} < (2m+2)u_{2m+1} < \epsilon$ ... ... (ii)

From (i) and (ii) it follows that $nu_n < \epsilon$ for all $n \geq 2m$.

This shows that $\lim nu_n = 0$.

Note. If (u_n) be a monotone decreasing sequence of positive real numbers and $\lim nu_n = 0$, then Σu_n is not necessarily convergent.

For example, let $u_n = \frac{1}{n \log n}, n > 1$. Then $u_{n+1} < u_n$ for all $n > 1$ and $\lim nu_n = 0$. But $\sum_{2}^{\infty} u_n$ is a divergent series.

Worked Examples (continued).

19. Prove that the series $(\frac{1}{2})^p + (\frac{1.3}{2.4})^p + (\frac{1.3.5}{2.4.6})^p + \cdots$ is convergent for $p > 2$ and divergent for $p \leq 2$.

Let $\sum_{n=1}^{\infty} u_n$ be the given series.

$$\begin{aligned} \text{Then } \frac{u_n}{u_{n+1}} = (\frac{2n+2}{2n+1})^p &= (1 + \frac{1}{n})^p(1 + \frac{1}{2n})^{-p} \\ &= \{1 + \frac{p}{n} + O(\frac{1}{n^2})\}\{1 - \frac{p}{2n} + O(\frac{1}{n^2})\} \\ &= 1 + \frac{p}{2n} + O(\frac{1}{n^2}). \end{aligned}$$

By Gauss's test, the series Σu_n is convergent if $\frac{p}{2} > 1$, i.e., if $p > 2$ and divergent if $\frac{p}{2} \leq 1$, i.e., if $p \leq 2$.

20. If Σu_n be a divergent series of positive real numbers prove that the series $\Sigma \frac{u_n}{1+u_n}$ is divergent.

Let $s_n = u_1 + u_2 + \cdots + u_n$.

Since the series Σu_n is a divergent series of positive real numbers, the sequence (s_n) is a monotone increasing sequence and $\lim s_n = \infty$.

Therefore for every natural number n we can choose a natural number p such that $s_{n+p} > 1 + 2s_n$.

Proof. (i) Let us choose a positive ϵ such that $l+\epsilon<1$.
There exists a natural number m such that
$l-\epsilon<\frac{|u_{n+1}|}{|u_n|}<l+\epsilon$ for all $n\geq m$.

Let $l+\epsilon=r$. Then $0<r<1$ and $\frac{|u_{n+1}|}{|u_n|}<r$ for all $n\geq m$.
Then $\frac{|u_{m+1}|}{|u_m|}<r, \frac{|u_{m+2}|}{|u_{m+1}|}<r, \cdots, \frac{|u_n|}{|u_{n-1}|}<r$.
Consequently, $\frac{|u_n|}{|u_m|}<r^{n-m}$ for all $n>m$
or, $|u_n|<\frac{|u_m|}{r^m}r^n$ for all $n>m$.
But Σr^n is a convergent series, since $0<r<1$.
By Comparison test, the series $\Sigma|u_n|$ is convergent. Therefore the series Σu_n is absolutely convergent.

(ii) Let us choose a positive ϵ such that $l-\epsilon>1$.
There exists a natural number k such that
$l-\epsilon<\frac{|u_{n+1}|}{|u_n|}<l+\epsilon$ for all $n\geq k$.

Therefore $\frac{|u_{n+1}|}{|u_n|}>l-\epsilon>1$ for all $n\geq k$.
Hence the sequence $(|u_n|)$ is ultimately a monotone increasing sequence of positive real numbers. So $\lim|u_n|\neq 0$ and this implies $\lim u_n\neq 0$. Consequently, the series Σu_n is divergent.

Theorem 6.4.4. Root test.

Let Σu_n be a series of arbitrary terms and let $\lim|u_n|^{1/n}=l$.
Then (i) Σu_n is absolutely convergent if $l<1$,
(ii) Σu_n is divergent if $l>1$.

Proof. (i) Let us choose a positive ϵ such that $l+\epsilon<1$.
There exists a natural number m such that
$l-\epsilon<|u_n|^{1/n}<l+\epsilon$ for all $n\geq m$.

Let $l+\epsilon=r$. Then $0<r<1$.
We have $|u_n|^{1/n}<r$ for all $n\geq m$.
or, $|u_n|<r^n$ for all $n\geq m$.
But Σr^n is a convergent series, since $0<r<1$.
By Comparison test, the series $\Sigma\mid u_n\mid$ is convergent. Therefore the series Σu_n is absolutely convergent.

(ii) Let us choose a positive ϵ such that $l-\epsilon>1$.
There exists a natural number k such that
$l-\epsilon<|u_n|^{1/n}<l+\epsilon$ for all $n\geq k$.
Therefore $|u_n|>1$ for all $n\geq k$.
So $\lim|u_n|\neq 0$ and this implies $\lim u_n\neq 0$. Consequently, the series Σu_n is divergent.

Worked Examples (continued).

2. Examine the convergence of the series
$1-\frac{2^2}{2!}+\frac{3^3}{3!}-\frac{4^4}{4!}+\cdots\cdots$

Let $\sum_{n=1}^{\infty}u_n$ be the given series. Then $u_n=(-1)^{n+1}\frac{n^n}{n!}$.
$\frac{|u_{n+1}|}{|u_n|}=\frac{(n+1)^{n+1}}{n+1}\cdot\frac{1}{n^n}=(1+\frac{1}{n})^n$ and $\lim\frac{|u_{n+1}|}{|u_n|}=e>1$.
By Ratio test, the series Σu_n is divergent.

3. Examine the convergence of the series $1-\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}-\cdots\cdots$

Let $\sum_{n=1}^{\infty}u_n$ be the given series. Then $u_n=(-1)^{n+1}\frac{(n!)^2}{(2n)!}$ for $n\geq 2$.
$\lim\frac{|u_{n+1}|}{|u_n|}=\lim\frac{(n+1)^2}{(2n+1)(2n+2)}=\frac{1}{4}<1$.
By Ratio test, the series Σu_n is absolutely convergent.

Alternating series.

Definition. If $u_n>0$ for all n, the series $\sum_{1}^{\infty}(-1)^{n+1}u_n$ called an *alternating series.*

Theorem 6.4.5. Leibnitz's test.

If (u_n) be a monotone decreasing sequence of positive real numbers and $\lim u_n=0$, then the alternating series
$u_1-u_2+u_3-u_4+\cdots$ is convergent.

Proof. Let $s_n=u_1-u_2+u_3-\cdots+(-1)^{n+1}u_n$.
Then $s_{2n+2}-s_{2n}=u_{2n+1}-u_{2n+2}\geq 0$ for all $n\in\mathbb{N}$.
The sequence (s_{2n}) is a monotone increasing sequence.
$s_{2n+1}-s_{2n-1}=-u_{2n}+u_{2n+1}\leq 0$ for all $n\in\mathbb{N}$.
The sequence (s_{2n+1}) is a monotone decreasing sequence.
Again $s_{2n}=u_1-u_2+u_3-u_4+\cdots-u_{2n}$
$=u_1-(u_2-u_3)-(u_4-u_5)-\cdots-u_{2n}<u_1$
The sequence (s_{2n}) is bounded above.
$s_{2n+1}=u_1-u_2+u_3-u_4+\cdots+u_{2n+1}$
$=(u_1-u_2)+(u_3-u_4)+\cdots+u_{2n+1}>u_1-u_2$
The sequence (s_{2n+1}) is bounded below.
Therefore both the sequences (s_{2n}) and (s_{2n+1}) are convergent. Because $\lim(s_{2n+1}-s_{2n})=\lim u_{2n+1}=0$, both the sequences (s_{2n+1}) and (s_{2n}) converge to the same limit.
Hence the sequence (s_n) is convergent and consequently the series $\Sigma(-1)^{n+1}u_n$ is convergent.

Note. If s be the sum of the series and s_n be the nth partial sum then $0 < (-1)^n(s - s_n) < u_{n+1}$ for all $n \in \mathbb{N}$.

$$s - s_n = (-1)^{n+2}[u_{n+1} - u_{n+2} + u_{n+3} - \cdots]$$
$$\text{or, } (-1)^n(s - s_n) = u_{n+1} - u_{n+2} + u_{n+3} - \cdots$$
$$= u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots < u_{n+1}.$$

Also $(-1)^n(s - s_n) = (u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + \cdots > 0$.

Combining, we have $0 < (-1)^n(s - s_n) < u_{n+1}$ for all $n \in \mathbb{N}$.

Examples.

1. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent by Leibnitz's test.

2. The series $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \cdots$ is convergent by Leibnitz's test.

3. The series $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \cdots$ is convergent by Leibnitz's test, since $\lim \frac{1.3.....(2n-1)}{2.4.....2n} = 0$ and $\frac{1.3.....(2n-1)}{2.4.....2n} > \frac{1.3.....(2n+1)}{2.4.....(2n+2)}$.

Theorem 6.4.6. Abel's test.

If the sequence (b_n) is a monotone bounded sequence and Σa_n is a convergent series then the series $\Sigma a_n b_n$ is convergent.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$, $t_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

Then $t_n = s_1b_1 + (s_2 - s_1)b_2 + (s_3 - s_2)b_3 + \cdots + (s_n - s_{n-1})b_n$
$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_n(b_n - b_{n+1}) + s_nb_{n+1}$.

Since Σa_n is convergent, the sequence (s_n) is convergent. Since the sequence (b_n) is monotonic and bounded, (b_n) is convergent.

Therefore s_nb_{n+1} tends to a limit ... ... (i)

Let $d_n = b_n - b_{n+1}$. Then either $d_n \geq 0$ for all n, or ≤ 0 for all n; and $d_1 + d_2 + \cdots + d_n = b_1 - b_{n+1}$ tends to a definite limit, since the sequence (b_n) is convergent. Therefore Σd_n is absolutely convergent.

Because the sequence (s_n) is bounded and the series Σd_n is absolutely convergent, the series Σs_nd_n is absolutely convergent by Theorem 6.4.2.

Therefore the sequence $(\sum\limits_{1}^{n} s_nd_n)$ is convergent ... ... (ii)

From (i) and (ii) it follows that the sequence (t_n) is convergent and this proves that the series Σa_nb_n is convergent.

Examples.

1. The series $\sum\limits_{2}^{\infty} \frac{(-1)^{n+1}}{n \log n}$ is convergent by Abel's test, since $\Sigma \frac{(-1)^{n+1}}{n}$ is a convergent series and the sequence $(\frac{1}{\log n})_2^{\infty}$ is a monotone decreasing sequence bounded below.

2. The series $\sum\limits_{1}^{\infty} \frac{(-1)^{n+1}.n^n}{(n+1)^{n+1}}$ is convergent by Abel's test, since $\Sigma \frac{(-1)^{n+1}}{n+1}$ is a convergent series and the sequence $((1+\frac{1}{n})^{-n})$ is a monotone decreasing sequence bounded below.

Theorem 6.4.7. Dirichlet's test.

If the sequence (b_n) is a monotone sequence converging to 0 and the sequence of partial sums (s_n) of the series Σa_n is bounded, then the series Σa_nb_n is convergent.

Proof. $s_n = a_1 + a_2 + \cdots + a_n$.

Let $t_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

Then $t_n = s_1b_1 + (s_2 - s_1)b_2 + (s_3 - s_2)b_3 + \cdots + (s_n - s_{n-1})b_n$
$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_n(b_n - b_{n+1}) + s_nb_{n+1}$.

Since the sequence (s_n) is bounded and $\lim b_n = 0$, $\lim s_nb_{n+1} = 0$.

Let $d_n = b_n - b_{n+1}$. Then either $d_n \geq 0$ for all n, or ≤ 0 for all n; and $d_1 + d_2 + \cdots + d_n (= b_1 - b_{n+1})$ tends to a definite limit since $\lim b_n = 0$. Therefore Σd_n is absolutely convergent.

Since the sequence (s_n) is bounded and the series Σd_n is absolutely convergent, the series Σs_nd_n is absolutely convergent by Theorem 6.4.2.

Hence the series Σs_nd_n is convergent and therefore the sequence (t_n) is convergent.

This proves that the series Σa_nb_n is convergent.

Note. Leibnitz's test is a particular case of Dirichlet's test. If (b_n) is a monotone decreasing sequence converging to 0, then the series $\sum\limits_{1}^{\infty}(-1)^{n+1}b_n$ is convergent by Dirichlet's test, since the sequence of partial sums (s_n) of the series $\Sigma(-1)^{n+1}$ is bounded.

This is Leibnitz's test for an alternating series.

Examples.

1. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is convergent by Dirichlet's test, since the sequence of partial sums (s_n) of the series $\Sigma(-1)^{n+1}$ is bounded and the sequence $(\frac{1}{\sqrt{n}})$ is a monotone decreasing sequence converging to 0.

2. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ is convergent by Dirichlet's test, since the sequence of partial sums (s_n) of the series $\Sigma(-1)^{n+1}$ is bounded and the sequence $(\frac{1}{\log(n+1)})$ is a monotone decreasing sequence converging to 0.

6.5. Conditionally convergent series.

Definition. A series Σu_n is called *conditionally convergent* if Σu_n is convergent but $\Sigma \mid u_n \mid$ is not convergent.

A conditionally convergent series is also called a *semi convergent* series or a *non-absolutely convergent* series.

Examples.

1. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is convergent, but the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ is divergent.
Therefore the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is conditionally convergent.

2. The series $\sum\limits_{1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$ is convergent, by Leibnitz's test.

But $\sum\limits_{1}^{\infty}\frac{1}{2n-1}$ is a divergent series. Therefore $\sum\limits_{1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$ is a conditionally convergent series.

3. The series $\sum\limits_{2}^{\infty}\frac{(-1)^{n+1}}{n\log n}$ is convergent, by Abel's test.

But the series $\sum\limits_{2}^{\infty}\frac{1}{n\log n}$ is divergent. Therefore $\sum\limits_{2}^{\infty}\frac{(-1)^{n+1}}{n\log n}$ is a conditionally convergent series.

4. Show that the series $\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots \cdots, a > 0$ is

(i) absolutely convergent if $p > 1$,
(ii) conditionally convergent if $0 < p \leq 1$.

Let Σu_n be the given series and $v_n = \mid u_n \mid$.
Then Σv_n is a series of positive real numbers and $v_n = \frac{1}{(n+a)^p}$.
Let $w_n = \frac{1}{n^p}$. Then $\lim \frac{v_n}{w_n} = 1$.
By Comparison test, Σv_n is convergent if $p > 1$,
Σv_n is divergent if $0 < p \leq 1$.

(i) $p > 1$. In this case Σu_n is an alternating series and $\Sigma \mid u_n \mid$ is convergent. Therefore Σu_n is absolutely convergent.

(ii) $0 < p \leq 1$. In this case (v_n) is a monotone decreasing sequence of positive real numbers and $\lim v_n = 0$. By Leibnitz's test, $\Sigma(-1)^{n+1}v_n$, i.e., Σu_n is convergent.

Since $\Sigma \mid u_n \mid$ is divergent, Σu_n is conditionally convergent.

Let Σu_n be a series of positive real numbers and let
$p_n = u_n$ if $u_n > 0$ $\quad q_n = 0$ if $u_n \geq 0$
$\quad = 0$ if $u_n \leq 0$; $\quad = u_n$ if $u_n < 0$.

Then Σp_n is a series of positive real numbers along with some 0's and Σq_n is a series of negative real numbers along with some 0's.

For example, for the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$
$\Sigma p_n = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$ and $\Sigma q_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \cdots$

$p_n = \frac{u_n + |u_n|}{2}$, $q_n = \frac{u_n - |u_n|}{2}$ and $u_n = p_n + q_n$.

Theorem 6.5.1. Let Σu_n be a series of arbitrary terms and $p_n = \frac{u_n+|u_n|}{2}$, $q_n = \frac{u_n-|u_n|}{2}$.

(i) If Σu_n is absolutely convergent, then both Σp_n and Σq_n are convergent.

(ii) If Σu_n is conditionally convergent, then both Σp_n and Σq_n are divergent.

Proof. (i) Since Σu_n is absolutely convergent, both Σu_n and $\Sigma|u_n|$ are convergent. But $p_n = \frac{u_n+|u_n|}{2}$, $q_n = \frac{u_n-|u_n|}{2}$. Hence Σp_n and Σq_n are both convergent.

(ii) Since Σu_n is conditionally convergent, Σu_n is convergent but $\Sigma|u_n|$ is divergent.

Now $|u_n| = 2p_n - u_n$ ... ... (A)

If we assume that Σp_n is convergent, then if follows from (A) that $\Sigma|u_n|$ is convergent, a contradiction. Therefore Σp_n is divergent.

Again $|u_n| = u_n - 2q_n$ ... ... (B)

If we assume that Σq_n is convergent, then if follows from (B) that $\Sigma|u_n|$ is convergent, a contradiction. Therefore Σq_n is divergent.

Note. For all n, $p_n \geq 0$ and $q_n \leq 0$. From (i) it follows that in an absolutely convergent series, the series formed by the positive terms alone and the series formed by the negative terms alone are both convergent.

From (ii) it follows that in a conditionally convergent series, the series formed by the positive terms alone and the series formed by the negative terms alone are both divergent.

Introduction and removal of brackets.

Theorem 6.5.2. Let Σu_n be a series of positive and negative real numbers and Σv_n is obtained from Σu_n by grouping it terms. Then

(i) if Σu_n converges to the sum s, then Σv_n also converges to s,

(ii) if Σv_n converges, then Σu_n may not be convergent.

Proof. (i) Let $v_1 = u_1 + u_2 + \cdots + u_{r_1}, v_2 = u_{r_1+1} + u_{r_1+2} + \cdots + u_{r_2}, \cdots, v_n = u_{r_{n-1}+1} + u_{r_{n-1}+2} + \cdots + u_{r_n}, \cdots \cdots$

Then (r_n) is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.

Then $t_n = u_1 + u_2 + \cdots + u_{r_n} = s_{r_n}$.

Since Σu_n converges to the sum s, $\lim s_n = s$.

The sequence (t_n) is a subsequence of the sequence (s_n) and therefore the sequence (t_n) also converges to the sum s.

In other words, the series Σv_n converges to the sum s.

(ii) That the converse is not true can be established by the following example.

Let $u_n = (-1)^{n+1}$. Then the series Σu_n is not convergent.

Let Σv_n be obtained from Σu_n by grouping the terms as $(1-1)+(1-1)+(1-1)+\cdots\cdots$

Then Σv_n is clearly a convergent series.

Examples.

1. Prove that $\log 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, by Leibnitz's test.

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ when $-1 < x \leq 1$.

So $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

Grouping the terms of the series as $(1-\frac{1}{2}) + (\frac{1}{3}-\frac{1}{4}) + (\frac{1}{3}-\frac{1}{6}) + \cdots$ we have the series $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

By Theorem 6.5.2, the sum of the series is $\log 2$.

2. Prove that $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots$

The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is convergent by Leibnitz's test.

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, when $-1 \leq x \leq 1$ (Gregory's series)

So $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Grouping the terms of the series as $(1-\frac{1}{3}) + (\frac{1}{5}-\frac{1}{7}) + \cdots$ we have the series $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \cdots$

By Theorem 6.5.2, the sum of the series is $\frac{\pi}{4}$.

Hence $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots = \frac{\pi}{8}$.

Re-arrangement of terms.

Theorem 6.5.3. *If the terms of an absolutely convergent series be rearranged, the series remains convergent and its sum remains unaltered.*

Proof. Let Σu_n be an absolutely convergent series and let the terms be re-arranged in any manner.

Let the new series be Σv_n. Then every u is a v and every v is a u.

Let $\Sigma|u_n| = s$. Then $\Sigma|u_n| + u_n$ is a series of positive real numbers and $u_n + |u_n| \leq 2|u_n|$.

By Comparison test, $\Sigma(|u_n| + u_n)$ is convergent.

Let $\Sigma(|u_n| + u_n) = s'$. Then $\Sigma u_n = s' - s$.

Since $\Sigma|u_n|$ and $\Sigma|u_n| + u_n)$ are convergent series of positive real numbers, their sums are not altered by re-arrangement of terms.

Therefore $\Sigma|v_n| = s$ and $\Sigma(|v_n| + v_n) = s'$.

Consequently, $\Sigma v_n = s' - s$. This shows that Σv_n is convergent and $\Sigma v_n = \Sigma u_n$. This proves the theorem.

We state here, without proof, an important theorem of Riemann about the behaviour of a conditionally convergent series.

Theorem 6.5.4. Riemann's theorem.

By appropriate re-arrangement of terms, a conditionally convergent series Σu_n can be made

(i) to converge to any number l, or (ii) to diverge to $+\infty$, or

(iii) to diverge to $-\infty$, or (iv) to oscillate finitely, or

(v) to oscillate infinitely.

Worked Example.

Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\log 2$, but the re-arranged series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots\cdots$ converges to $\frac{1}{2}\log 2$.

We have $\lim_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n) = \gamma$.

Let $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n = \gamma_n$. Then $\lim_{n\to\infty} \gamma_n = \gamma$.

Let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$.

Then $s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$

$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}) - 2(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n})$

$= \log 2n + \gamma_{2n} - (1 + \frac{1}{2} + \cdots + \frac{1}{n})$

$= \log 2n + \gamma_{2n} - (\log n + \gamma_n) = \log 2 + \gamma_{2n} - \gamma_n$.

Therefore $\lim s_{2n} = \log 2$.

$s_{2n+1} = s_{2n} + \frac{1}{2n+1}$. Therefore $\lim s_{2n+1} = \lim s_{2n} = \log 2$.

the elements of the array in the form of an infinite series in two ways –

(i) $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (A)

(ii) $a_0b_0 + (a_1b_0 + a_1b_1 + a_0b_1) + (a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2) + \ldots$ ... (B)

Let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$, $t_n = b_0 + b_1 + b_2 + \cdots + b_n$. Then $\lim s_n = s$, $\lim t_n = t$.

Let σ_n be the sum of the first $(n+1)$ terms of the series (B). Then $\sigma_0 = s_0t_0$, $\sigma_1 = s_1t_1$, $\sigma_2 = s_2t_2, \ldots$ $\sigma_n = s_nt_n, \ldots$

$\lim \sigma_n = \lim s_nt_n = st$. Therefore the series (B) is convergent and has the sum st.

Since the series (B) is a convergent series of positive terms, the series remains convergent with the same sum st after removal of brackets. The series (A) is obtained from the resulting series by rearrangement of terms and then by introduction of brackets. Hence the series (A) remains convergent with the same sum st.
This completes the proof.

Theorem 6.6.2. If $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$ be two absolutely convergent series with s and t as their sums, then the series $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (i.e., the Cauchy product) is absolutely convergent and has the sum st.

Proof. Since an absolutely convergent series remains convergent either by rearrangement of terms or by introduction of brackets and in either case the sum remains unaltered, the theorem can be established by following the same lines of proof as discussed in the previous theorem.

The following theorem due to Mertens is a further extension of the previous one and it is stated below without proof.

Theorem 6.6.3. Mertens' theorem.

If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ be convergent series with sums s and t respectively and one of the series, say $\sum\limits_{n=0}^{\infty} a_n$ be absolutely convergent, then the series $\sum\limits_{n=0}^{\infty} c_n$, where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$, is convergent and its sum is st.

Note. If both the series $\sum a_n$ and $\sum b_n$ be non-absolutely convergent, then their Cauchy product may not be convergent.

For example, let us consider the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$.

The series is non-absolutely convergent. Let the series be $\sum\limits_{n=1}^{\infty} a_n$.

Let the Cauchy product of tne series $\sum\limits_{n=1}^{\infty} a_n . \sum\limits_{n=1}^{\infty} a_n$ be $\sum\limits_{n=1}^{\infty} c_n$.

Then $c_n = (-1)^{n-1}[\frac{1}{\sqrt{1.n}} + \frac{1}{\sqrt{2(n-1}} + \frac{1}{\sqrt{3(n-2}} + \cdots + \frac{1}{\sqrt{n.1}}]$.

$r(n-r+1) = (\frac{n+1}{2})^2 - (\frac{n+1}{2} - r)^2 \le (\frac{n+1}{2})^2$ for all r satisfying $1 \le r \le n$.

$|c_n| \ge \frac{2n}{n+1}$ and this implies $\lim c_n \ne 0$. The necessary condition for convergence of the series $\sum\limits_{n=1}^{\infty} c_n$ is not satisfied.

This establishes that the Cauchy product of two non-absolutely convergent series may not be convergent.

If, however, the Cauchy product of two non-absolutely convergent series be convergent, then the following theorem due to Abel establishes that the sum of the Cauchy product is the product of the sums of the series.

Theorem 6.6.4. Abel's theorem.

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be convergent series with sums A and B respectively and if their Cauchy product $\sum\limits_{n=1}^{\infty} c_n$ be convergent with the sum C, then $C = AB$.

First we prove the following lemma.

Lemma. If $\lim u_n = u, \lim v_n = v$, then $\lim \frac{u_nv_1 + u_{n-1}v_2 + \cdots + u_1v_n}{n} = uv$.

Proof. Let $u_n = u + \alpha_n$, $v_n = v + \beta_n$. Then $\lim \alpha_n = 0, \lim \beta_n = 0$.

$\frac{u_nv_1 + u_{n-1}v_2 + \cdots + u_1v_n}{n} = \frac{(u+\alpha_n)(v+\beta_1)+(u+\alpha_{n-1})(v+\beta_2)+\cdots+(u+\alpha_1)(v+\beta_n)}{n}$
$= uv + \frac{u}{n}[\beta_1 + \beta_2 + \cdots + \beta_n] + \frac{v}{n}[\alpha_1 + \alpha_2 + \cdots + \alpha_n] + \frac{\alpha_n\beta_1 + \alpha_{n-1}\beta_2 + \cdots + \alpha_1\beta_n}{n}$
... (i)

By Cauchy's theorem, $\lim \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = 0, \lim \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} = 0$.

Since $\lim \alpha_n = 0$, the sequence (α_n) is bounded. So there exists a positive real number k such that $\alpha_n < k$ for all n.

Therefore $\lim \frac{\alpha_n\beta_1 + \alpha_{n-1}\beta_2 + \cdots + \alpha_1\beta_n}{n} \le k. \lim \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} = 0$.

From (i) it follows that $\lim \frac{u_nv_1 + u_{n-1}v_2 + \cdots + u_1v_n}{n} = uv$.