CHAPTER I

BASIC CONCEPTS OF DIFFERENTIAL EQUATIONS

1.1. Introduction.

A relation connecting an independent variable x, a dependent variable y and one or more of their differential coefficients or differentials is called a *differential equation*. For examples,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)y = 0, \quad \cot y\, dx - \tan x . dy = 0.$$

In fact, a differential equation as given above which involves only one independent variable is called *ordinary*, while those involving more than one independent variables are called *partial* differential equations. Partial differential equations will involve partial derivatives. For examples,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial y} = 0.$$

There are differential equations, which do not contain the variables explicitly. For examples,

$$\frac{dy}{dx} = 5, \quad x\frac{d^2y}{dx^2} = 4.$$

Many of the general laws of Physics, Chemistry, Biology and Astronomy find their natural expressions in the form of differential equations.

A total differential equation contains two or more dependent variables together with their differentials or differential coefficients with respect to a single independent variable. This may, or may not, occur explicitly in the equation. In the total differential equation

$$u\, dx + v\, dy + w\, dz = 0,$$

where u, v, w are functions of x, y and z, any one of the variables may be regarded as independent and the others as dependent. Thus, taking x as independent variable, the above equation may be written as

$$u + v\frac{dy}{dx} + w\frac{dz}{dx} = 0.$$

The *order* of a differential equation is the order of the highest ordered differential coefficient involving in it while the *degree* of an equation is the greatest exponent of the highest ordered derivative when the equation has been made rational and integral as far as the derivatives are concerned. Thus

$\frac{dy}{dx} + xy = x^2$ is an equation of first order and first degree ;

$\frac{d^2y}{dx^2} = \frac{x^2}{y(1+\sqrt{x})}$ is an equation of second order and first degree ;

$\left(\frac{d^2y}{dx^2}\right)^2 = 4$ is an equation of second order and second degree ;

$x^2 = y^2\left(\frac{dy}{dx}\right)^2$ is an equation of first order and second degree.

Let us consider the equation $\frac{d^2y}{dx^2} = 5\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$.

This equation is to be squared to rationalise it and then it can be easily seen that the greatest exponent of the highest ordered derivative $\frac{d^2y}{dx^2}$ is two. Hence the equation is of second order and second degree. It should be noted that the determination of the degree does not require the variables x and y to be made rational and integral.

When, in an ordinary or partial differential equation, the dependent variable and its derivatives occur to the first degree only, and not as higher powers or products, the equation is called *linear* ; otherwise it is *non-linear*. The coefficients of a linear equation are, therefore, either constants or functions of the independent variable or variables. For examples, $\frac{d^2y}{dx^2} + y = x$, $x^3\frac{d^2y}{dx^2} + (\cos x)\frac{dy}{dx} + (\sin x)y = 0$ are ordinary linear differential equations of the second order while the equations

$$(x+y)^2\frac{dy}{dx} = a, \quad \frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + x(\sin y) = 0$$

are ordinary non-linear equations of the first and second order respectively.

Any relation connecting the dependent as well as the independent variables will be called the *solution or primitive* of the differential equation, if it reduces the differential equation to an identity when substituted in it. The solution of the differential equation does not contain any derivative. Thus

$y = 2x$ is a solution of the differential equation $\frac{dy}{dx} = 2$.

1.2. Formation of differential equations.

Consider the relation $y = cx$, where c is an arbitrary constant. Differentiating both sides of this, with respect to x, we get

$$\frac{dy}{dx} = c.$$

Eliminating the arbitrary constant from these two, we get the ordinary differential equation

$$\frac{dy}{dx} = \frac{y}{x}, \qquad \ldots \quad (1)$$

which is of first order and first degree. Substituting $y = cx$ in (1), we see that the equation is identically satisfied, showing that $y = cx$ is a solution of the first ordered equation (1). We notice further that the solution of a differential equation of first order and first degree will involve one arbitrary constant.

In a similar manner, differentiating both sides of the relation

$$y = A\cos(x + B), \qquad \ldots \quad (2)$$

where A and B are arbitrary constants, twice with respect to x, we get

$$\frac{dy}{dx} = -A\sin(x + B) \qquad \ldots \quad (3)$$

and
$$\frac{d^2y}{dx^2} = -A\cos(x + B). \qquad \ldots \quad (4)$$

Eliminating the arbitrary constants A and B from (2) and (4), we get the second ordered ordinary differential equation

$$\frac{d^2y}{dx^2} + y = 0. \qquad \ldots \quad (5)$$

Thus (2) is a solution of the equation (5) ; for, (2), when substituted in (5), reduces it to an identity. As before, we observe that the solution of a differential equation of second order involves two arbitrary constants.

1.5. Cauchy's problem.

Consider the first ordered differential equation $\frac{dy}{dx} = f(x, y)$, whose general solution contains one arbitrary constant.

It is sufficient to specify the value y_0 of the particular solution for some value x_0 of the independent variable x, that is, to find a point (x_0, y_0) through which the integral curve of the given equation must pass.

But this is not sufficient for equations of higher order. For instance the general solution of the equation $\frac{d^2y}{dx^2} = 0$ is

$$y = Ax + B,$$

where A and B are arbitrary constants.

The equation $y = Ax + B$ defines a two-parameter family of straight lines on the xy-plane. To specify a definite straight line, it is not sufficient to specify a point (x_0, y_0) through which the line must pass. It is also necessary to specify the slope of the line at the point (x_0, y_0) as given by

$$\frac{dy}{dx} \text{ at } x = x_0, \text{ that is, } \left(\frac{dy}{dx}\right)_0.$$

In the general case of the n-th ordered differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0, \quad \ldots (1)$$

in order to isolate a particular solution, we must have n conditions

$$y = y_0, \frac{dy}{dx} = \left(\frac{dy}{dx}\right)_0, \ldots\ldots, \frac{d^{n-1}y}{dx^{n-1}} = \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_0, \text{ at } x = x_0 \quad \ldots (2)$$

in which $y_0, \left(\frac{dy}{dx}\right)_0, \ldots\ldots, \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_0$ are given numbers.

Cauchy's problem is to find a solution of the differential equation (1) which satisfies the specific conditions (2).

1.6. Illustrative Examples.

Ex. 1. *Eliminate the arbitrary constants A and B from the relation.*

$$y = Ae^{x} + Be^{-x} + x^{2}.$$

From the given relation, we have

$$\frac{dy}{dx} = Ae^{x} - Be^{-x} + 2x$$

and $$\frac{d^2y}{dx^2} = Ae^{x} + Be^{-x} + 2 = y - x^2 + 2.$$

Therefore $$\frac{d^2y}{dx^2} - y = 2 - x^2.$$

This is the required eliminant.

Ex. 2. *Show that the differential equation satisfied by the family of curves given by $c^2 + 2cy - x^2 + 1 = 0$, where c is the parameter of the family, is*

$$(1 - x^2)p^2 + 2xyp + x^2 = 0, \text{ where } p = \frac{dy}{dx}.$$ [C. H. 1982]

From the given equation of the family of curves, we have, on differentiation with respect to x,

$$2cp - 2x = 0.$$

Therefore $$c = \frac{x}{p}.$$

Eliminating c from the given equation, we get the corresponding differential equation as

$$\frac{x^2}{p^2} + \frac{2xy}{p} - x^2 + 1 = 0$$

or, $$(1 - x^2)p^2 + 2xyp + x^2 = 0.$$

Ex. 3. *Obtain the differential equation of the system of confocal conics*

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

in which λ is the arbitrary parameter and a, b are given constants.

Let us eliminate λ from the given equation and its first derived equation

$$\frac{2x}{a^2 + \lambda} + \frac{2yy'}{b^2 + \lambda} = 0, \text{ where } y' = \frac{dy}{dx}.$$

From the last equation, we have

$$\frac{a^2+\lambda}{x} = \frac{b^2+\lambda}{-yy'} = k \text{ (say)}. \qquad \ldots (1$$

Putting this in the given equation, we have

$$\frac{x^2}{kx} + \frac{y^2}{-kyy'} = 1$$

$$\text{or,} \qquad k = -\frac{1}{y'}(y - xy').$$

Substituting this value of k in (1), we get

$$a^2 + \lambda = kx = \frac{x^2y' - yx}{y'}$$

$$\text{and} \qquad b^2 + \lambda = -kyy' = y(y - xy').$$

Subtracting, we have

$$a^2 - b^2 = \frac{x^2y' - yx}{y'} + (xy' - y)y.$$

$$\text{Therefore} \qquad (a^2 - b^2)y' = x(xy' - y) + yy'(xy' - y)$$

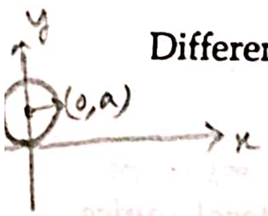
$$= (xy' - y)(x + yy').$$

This is the required differential equation, whose primitive is the give system of confocal conics.

Ex. 4. *Obtain the differential equation of all circles each of which touche the axis of x at the origin.* [C. H. 1985 ; V. H. 1988

The equation of the circles touching the x-axis at the origin is

$x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant.



Differentiating both sides with respect to x, we get

$$2x + 2yy' - 2ay' = 0, \text{ where } y' = \frac{dy}{dx}$$

$$\text{or,} \quad a = \frac{x + yy'}{y'}.$$

Putting for a in the equation, we have the required differential equation as

$$x^2 + y^2 = \frac{2y(x + yy')}{y'}$$

$$\text{or,} \quad (x^2 + y^2)y' - 2y^2y' = 2xy$$

$$\text{or,} \quad (x^2 - y^2)y' = 2xy.$$

CHAPTER II

EQUATIONS OF FIRST ORDER AND FIRST DEGREE

2.1. Existence and uniqueness of solution.

We state the theorem* of *existence and uniqueness of solution* of ordinary differential equation of first order without proof.

Given a differential equation

$$\frac{dy}{dx} = f(x, y), \qquad \dots (1)$$

where $f(x, y)$ is subject to the following conditions :

(i) $f(x, y)$ is continuous in a given region G,

(ii) $|f(x, y)| \leq M$, a fixed real number in G,

(iii) $|f(x, y_1) - f(x, y_2)| \leq k\,|y_1 - y_2|$, k being a fixed quantity for any two points (x, y_1) and (x, y_2) in the region G.

If now (x_0, y_0) be any point in G such that the rectangle R as given by $|x - x_0| \leq a$, $|y - y_0| \leq b$, where $b > aM$ such that R lies wholly within G, then there exists one and only one continuous function $y = \phi(x)$ having continuous derivatives in $|x - x_0| \leq a$, which satisfies the differential equation (1) and takes up the value $\phi(x_0) = y_0$ when

$$x = x_0.$$

The condition (*iii*), that is, $|f(x, y_1) - f(x, y_2)| \leq k\,|y_1 - y_2|$ provided (x, y_1) and (x, y_2) are any two points in G, is known as *Cauchy-Lipschitz condition.* If $f(x, y)$ admits of continuous partial derivatives and hence $\left|\frac{\partial f(x, y)}{\partial y}\right| < k$, where k is fixed, then the Cauchy-Lipschitz condition is satisfied.

An ordinary differential equation of first order and first degree

$$\frac{dy}{dx} = f(x, y)$$

can always be written as $M\,dx + N\,dy = 0$,

* For proof of this theorem, see Appendix.

where M and N are functions of x and y. Assuming that the equation has a solution, we shall discuss methods by which the general solutions of these equations can be found in terms of known functions. We classify these equations according to the methods by which they are solved. These classifications are

(A) Equations solvable by separation of variables,

(B) Homogeneous equations,

(C) Exact equations,

(D) Linear equations.

2.2. Solution by separation of variables.

If the equation $M\,dx + N\,dy = 0$ can be put in the form

$$f_1(x)\,dx + f_2(y)\,dy = 0,$$

that is, in the separated variables form, then the equation can be solved easily by integrating each term separately. The general solution of the above equation is $\int f_1(x)\,dx + \int f_2(y)\,dy = c,$
where c is an arbitrary constant. By giving particular values to c, we shall get particular solutions.

The equations of the form

$$\frac{dy}{dx} = f(x)\,g(y) \quad \text{or} \quad f_1(x)\,\phi_1(y)\,dx + f_2(y)\,\phi_2(x)\,dy = 0$$

can also be put in the above form and integrated. Sometimes a transformation of the dependent variable will be used to facilitate separation of variables.

2.3. Illustrative Examples.

Ex. 1. *Solve* : $x^2\dfrac{dy}{dx} + y = 1.$

We have, from the given equation,

$$x^2\frac{dy}{dx} = 1 - y$$

or, $$\frac{dx}{x^2} = \frac{dy}{1-y}.$$

The variables have been separated.

Now, integrating both sides, we get the general solution as

$-\frac{1}{x} = -\log(1-y) + c_1$, where c_1 is an arbitrary constant

or, $\log\frac{1-y}{c} = \frac{1}{x}$, taking $c_1 = \log c$

or, $1 - y = c\,e^{\frac{1}{x}}$

or, $y = 1 - c\,e^{\frac{1}{x}}$.

Ex. 2. *Solve* : $x\sqrt{y}\,dx + (1+y)\sqrt{1+x}\,dy = 0$.

Dividing throughout by $\sqrt{y(1+x)}$, we get

$$\frac{x\,dx}{\sqrt{1+x}} + \frac{1+y}{\sqrt{y}}\,dy = 0$$

or, $\left(\sqrt{1+x} - \frac{1}{\sqrt{1+x}}\right)dx + \left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)dy = 0$.

Integrating, we get the general solution as

$\frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + 2\sqrt{y} + \frac{2}{3}y^{\frac{3}{2}} = c$, where c is an arbitrary constant

or, $(x-2)\sqrt{1+x} + (y+3)\sqrt{y} = k$, where $k = \frac{3c}{2}$.

Ex. 3. *Solve* : $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.

From the given equation, we have

$$y - ay^2 = (a+x)\frac{dy}{dx}$$

or, $\frac{dx}{a+x} = \frac{dy}{y(1-ay)} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$.

Integrating both sides, we get

$\log(a+x) = \log cy - \log(1-ay)$, where c is an arbitrary constant.

Thus the general solution is

$$(a+x)(1-ay) = cy.$$

Ex. 4. *Solve* : $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$
and find the particular solution, if $y = 0$ *when* $x = 1$.

If the equation $M\,dx + N\,dy = 0$ can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

that is, if M and N be homogeneous functions of x and y of same degree, then the equation is called *homogeneous.* In this case, the substitution of $y = vx$, where v is a function of x, enables us to separate the variables. By this substitution, we change the variable, such that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then, after integration, v is replaced by $\frac{y}{x}$.

Note. $\frac{dy}{dx} = f(x, y)$ is a homogenous equation, if $f(tx, ty) = f(x, y)$ for all t.

It can be easily verified that the equations

$$\frac{dy}{dx} = 3\log(x + y) - \log(x^3 + y^3) \text{ and } \frac{dy}{dx} = \frac{4y\sqrt{x} - 5x\sqrt{y}}{\sqrt{x}(-x + 7y)}$$

are homogeneous.

2.5. Non-homogeneous equation reducible to homogeneous form.

Consider a non-homogeneous equation of the form

$$(a_1x + b_1y + c_1)\,dx = (a_2x + b_2y + c_2)\,dy,$$

that is,
$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \qquad \ldots (1)$$

in which $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

This equation can be made homogeneous by the substitution

$$x = x' + h \text{ and } y = y' + k,$$

where h and k are constants and are so chosen that

$$a_1h + b_1k + c_1 = 0 \qquad \ldots (2)$$

and
$$a_2h + b_2k + c_2 = 0.$$

The relations (2) determine the constants h and k.

The equation (1) is then reduced to the homogeneous equation

$$\frac{dy'}{dx'} = \frac{a_1x' + b_1y'}{a_2x' + b_2y'},$$

which can be solved after the substitution $y' = vx'$ as before.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then the substitution

$$a_1 x + b_1 y = v,$$

that is, $$a_1 + b_1 \frac{dy}{dx} = \frac{dv}{dx}$$

will transform the equation to a form, which can be easily solved.

2.6. Illustrative Examples.

Ex. 1. *Solve* : $x^2 y\, dx - (x^3 + y^3)\, dy = 0$.

Here M and N are homogeneous functions of x and y of degree 3.

The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}.$$

Let $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Then the equation becomes

$$v + x\frac{dv}{dx} = \frac{x^3 v}{x^3 + x^3 v^3} = \frac{v}{1 + v^3}$$

or, $$x\frac{dv}{dx} = \frac{v}{1+v^3} - v = -\frac{v^4}{1+v^3}$$

or, $$\frac{1+v^3}{v^4}\, dv = -\frac{dx}{x},$$

in which the variables v and x have been separated.

This gives $$\left(\frac{1}{v^4} + \frac{1}{v}\right) dv = -\frac{dx}{x}.$$

Integrating both sides, we get

$$-\frac{1}{3v^3} + \log v = -\log x + \log c,\ c \text{ being a constant}$$

or, $$\log\frac{vx}{c} = \frac{1}{3v^3}.$$

Therefore $$\frac{vx}{c} = e^{\frac{1}{3v^3}} = e^{\frac{x^3}{3y^3}}.$$

Thus the solution is $$y = ce^{\frac{x^3}{3y^3}}.$$

2.7. Exact equation and its solution by inspection.

If the differential equation $M\,dx + N\,dy = 0$ can be expressed in the form $du = 0$, where u is a function of x and y, without multiplying by any factor, then the equation $M\,dx + N\,dy = 0$ is said to be an *exact differential equation* and its general solution is $u(x, y) = c$, where c is an arbitrary constant.

Some differential equations are easily integrated by mere inspection. Sometimes the equation is to be multiplied by some function of x and y, so that it can be integrated by putting it in an integrable form. We give below some typical examples of the differentials of some functions which will help us to find the primitive of the equation by inspection.

$$x\,dy + y\,dx = d\,(xy)\ ; \qquad \frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right);$$

$$\frac{x\,dy - y\,dx}{y^2} = -\,d\left(\frac{x}{y}\right); \qquad \frac{x\,dy - y\,dx}{xy} = d\left(\log\frac{y}{x}\right);$$

$$\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right); \qquad \frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right);$$

$$\frac{2xy\,dy - y^2\,dx}{x^2} = d\left(\frac{y^2}{x}\right); \qquad \frac{y^2\,dx + 2xy\,dy}{x^2y^4} = -\,d\left(\frac{1}{xy^2}\right);$$

$$\frac{x\,dy + y\,dx}{\sqrt{1 - x^2y^2}} = d\,\{\sin^{-1}(xy)\}.$$

2.8. General method of solution of exact equations.

We have stated earlier that the ordinary differential equation $M\,dx + N\,dy = 0$ will be exact, if there exists a function $u(x, y)$, such that $M\,dx + N\,dy = du$. We now establish the condition for that.

Theorem. *The necessary and sufficient condition for the ordinary differential equation $M\,dx + N\,dy = 0$ to be exact is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

[We assume here that the functions M and N have continuous partial derivatives.]

If the equation $M\,dx + N\,dy = 0$ ... (1)

be exact, then there must be a function u of x and y, such that $M\,dx + N\,dy =$ a total differential of $u = du$. ... (2)

Also, we have $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad \ldots (3)$$

x and y being independent variables.

Now the two expressions (2) and (3) for du are identical and hence, from (2) and (3), we shall have

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N.$$

Therefore $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}.$$

Hence $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

if the partial derivatives of M and N be continuous.

Thus the condition is *necessary.*

To prove that this condition is also *sufficient,* we are to show that if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $M\,dx + N\,dy = du$.

Let us put $P = \int M\,dx$, where, in the integrand, y is supposed to be a constant.

Then $\frac{\partial P}{\partial x} = M$ and we have

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 P}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right).$$

Therefore $N = \frac{\partial P}{\partial y} + f(y)$, where $f(y)$ is a function of y.

Using these values of M and N, we can write

$$M\,dx + N\,dy = \frac{\partial P}{\partial x}\,dx + \left\{\frac{\partial P}{\partial y} + f(y)\right\}dy$$

$$= d\{P + F(y)\}, \text{ where } dF(y) = f(y)\,dy\,.$$

Now, writing $P + F(y) = u(x, y)$, we have

$$M\,dx + N\,dy = du\,.$$

Thus, to solve an equation of the form $M\,dx + N\,dy = 0$, we are to arrange the terms in groups each of which is an exact differential, so that $u(x, y)$ may be obtained by inspection only. This method has been discussed earlier.

If this cannot be done, then we are to test the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for the exactness of the equation first.

If it be found to be exact, then, to determine the function $u(x, y)$, we use the relation

$$\frac{\partial u}{\partial x} = M,$$

which, on integration, gives $u = \int M\,dx + f(y)$, where $f(y)$ is a function of y.

Now, to determine the function $f(y)$, we equate the total differential of $u(x, y)$ to $(M\,dx + N\,dy)$.

We see that all the terms of $u(x, y)$ containing x must appear in $\int M\,dx$. Hence the differential of this integral with respect to y must have all terms of $N\,dy$ which contain x. Hence the rule for solving an exact equation of the form $M\,dx + N\,dy = 0$ is

Integrate the terms of $M\,dx$ considering y as constant ; then integrate those terms of $N\,dy$ which do not contain x and then equate the sum of these integrals to a constant.

Cor. *In the exact equation $M\,dx + N\,dy = 0$, if M and N be homogeneous functions of x and y of degree $n\ (\neq -1)$, then the primitive can be obtained without any integration and the primitive is*

$$Mx + Ny = \textit{constant}\,.$$

Since the equation is exact, we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. . . . (1)

Again, since M and N are homogeneous functions of degree n, we have, by Euler's theorem,

$$x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}=nM \qquad \ldots \qquad (2)$$

and
$$x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}=nN\,. \qquad \ldots \qquad (3)$$

Let $u=Mx+Ny$, so that we have

$$\frac{\partial u}{\partial x}=M+x\frac{\partial M}{\partial x}+y\frac{\partial N}{\partial x}=M+x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}, \text{ by (1)}$$

$$=M+nM, \text{ by (2)}$$
$$=(n+1)\,M.$$

Similarly, by (1) and (3), we get

$$\frac{\partial u}{\partial y}=(n+1)\,N\,.$$

Therefore $du=\dfrac{\partial u}{\partial x}dx+\dfrac{\partial u}{\partial y}dy=(n+1)\,(M\,dx+N\,dy)\,.$

Hence $M\,dx+N\,dy=\dfrac{1}{n+1}du=\dfrac{1}{n+1}d\,(Mx+Ny)\,,\, n\neq -1\,.$

Thus the primitive is $Mx+Ny=$ constant.

2.9. Integrating factors.

Sometimes it is seen that an equation, as it stands, is not exact but it can be made exact by multiplying it by some function of x and y. The function, which multiplies the equation to make it exact, is called *integrating factor.*

Let $M\,dx+N\,dy=0$ be an ordinary differential equation and
$$Mx\pm Ny=0\,.$$

We have $\dfrac{dy}{dx}=-\dfrac{M}{N}=\pm\dfrac{y}{x}$,

which can be integrated easily and in this situation no integrating factor is necessary.

Theorem. *The number of integrating factors of an equation* $M\,dx+N\,dy=0$*, which has a solution, is infinite.*

Let $\mu\,(x,y)$ be an integrating factor of the equation $M\,dx+N\,dy=0$, so that
$$\mu\,(M\,dx+N\,dy)=du\,.$$

Hence $u(x, y) = c$ is a solution of the equation.

If $f(u)$ be any function of u, then

$$\mu f(u)(M\,dx + N\,dy) = f(u)\,du.$$

Now, the right-hand expression is an exact differential, since $f(u)\,du$ can easily be integrated to give $\phi(u)$. Thus the solution of the equation is

$$\phi(u) = c,$$

showing that $\mu f(u)$ is also an integrating factor of the equation

$$M\,dx + N\,dy = 0.$$

Since $f(u)$ is an arbitrary function of u, the number of integrating factors is infinite.

2.10. Rules for finding integrating factors.

It is seen that an integrating factor can be found by inspection in simple cases. But in most cases when integrating factor cannot be found by inspection, the following rules are used to find it. For that, we consider the differential equation $M\,dx + N\,dy = 0$, in which

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Rule I. *If $Mx + Ny \neq 0$ and the equation be homogeneous, then $\dfrac{1}{Mx+Ny}$ is an integrating factor of the equation $M\,dx + N\,dy = 0$.*

We have $M\,dx + N\,dy$

$$= \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$= \frac{1}{2}\left[(Mx+Ny)\,d\left\{\log(xy)\right\}+(Mx-Ny)\,d\left(\log\frac{x}{y}\right)\right]. \quad \ldots \quad (1)$$

Since $Mx + Ny \neq 0$, we have, dividing both sides by $(Mx + Ny)$,

$$\frac{M\,dx+N\,dy}{Mx+Ny} = \frac{1}{2}\,d\left\{\log(xy)\right\}+\frac{1}{2}\,\frac{Mx-Ny}{Mx+Ny}\,d\left(\log\frac{x}{y}\right). \quad \ldots \quad (2)$$

Now, $(Mx + Ny)$ is homogeneous ; hence $\dfrac{Mx-Ny}{Mx+Ny}$ is also homogeneous, and is equal to a function of $\dfrac{x}{y}$, say $f\left(\dfrac{x}{y}\right)$.

Therefore (2) becomes

$$\frac{M\,dx+N\,dy}{Mx+Ny}=\tfrac{1}{2}d\left\{\log(xy)\right\}+\tfrac{1}{2}f\left(\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)$$

$$=\tfrac{1}{2}d\left\{\log(xy)\right\}+\tfrac{1}{2}F\left(\log\frac{x}{y}\right)d\left(\log\frac{x}{y}\right), \quad \ldots (3)$$

since $f\left(\frac{x}{y}\right)=f\left\{e^{\log\frac{x}{y}}\right\}=F\left(\log\frac{x}{y}\right)$.

The right-hand side of (3) is an exact differential.

Hence we see that $\frac{1}{Mx+Ny}$ is an integrating factor of the equation.

Rule II. *If $Mx-Ny\neq 0$ and the equation can be written as*

$$\{f(xy)\}y\,dx+\{F(xy)\}x\,dy=0,$$

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then $\frac{1}{Mx-Ny}$ *is an integrating factor of the equation.*

Since $Mx-Ny\neq 0$, dividing both sides of (1) by $(Mx-Ny)$, we get

$$\frac{M\,dx+N\,dy}{Mx-Ny}=\frac{1}{2}\frac{Mx+Ny}{Mx-Ny}d\{\log(xy)\}+\tfrac{1}{2}d\left(\log\frac{x}{y}\right).$$

Now, we have $M=\{f(xy)\}y$ and $N=\{F(xy)\}x$.

Therefore

$$\frac{M\,dx+N\,dy}{Mx-Ny}=\frac{1}{2}\frac{f(xy)+F(xy)}{f(xy)-F(xy)}d\{\log(xy)\}+\tfrac{1}{2}d\left(\log\frac{x}{y}\right)$$

$$=\tfrac{1}{2}\phi(xy)\,d\{\log(xy)\}+\tfrac{1}{2}d\left(\log\tfrac{x}{y}\right)$$

$$=\tfrac{1}{2}\psi\{\log(xy)\}\,d\{\log(xy)\}+\tfrac{1}{2}d\left(\log\frac{x}{y}\right), \quad \ldots (4)$$

since $\phi(xy)=\phi\{e^{\log(xy)}\}=\psi\{\log(xy)\}$.

The right-hand expression of (4) being an exact differential, $\frac{1}{Mx-Ny}$ is an integrating factor of the equation.

Note. If $Mx-Ny=0$ identically, then $\frac{M}{N}=\frac{y}{x}$ and the equation $M\,dx+N\,dy=0$ reduces to $x\,dy+y\,dx=0$, whose solution is $xy=c$.

Rule III. *If* $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ *be a function of x alone, say f(x), then* $e^{\int f(x)\,dx}$ *is an integrating factor of the equation.* 99,10

Let μ be an integrating factor of the equation $M\,dx + N\,dy = 0$,
so that $$(\mu M)\,dx + (\mu N)\,dy = 0$$
is an exact differential equation.

Hence the condition $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$ must be satisfied.

This gives $$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = 0. \quad \ldots \quad (5)$$

Now suppose μ is a function of x only, so that $\frac{\partial \mu}{\partial y} = 0$.

Then (5) gives
$$\frac{d\mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx. \quad \ldots \quad (6)$$

Now, since μ is a function of x alone, the right hand side of (6) is a function of x only.

Let us put $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = f(x)$,

so that (6) becomes $\frac{d\mu}{\mu} = f(x)\,dx$,

which gives $$\mu = e^{\int f(x)\,dx}$$

Thus $e^{\int f(x)\,dx}$ is an integrating factor of the equation.

Note. $e^{\int P\,dx}$ is an integrating factor of the equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only, since the equation can be written as $(Py - Q)\,dx + dy = 0$, so that $M = Py - Q$, $N = 1$ and $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = P$, which is a function of x alone.

Rule IV. *If* $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ *be a function of y alone, say* $\phi(y)$, *then* $e^{\int \phi(y)\,dy}$ *is an integrating factor of the equation.*

The proof is similar to that in Rule III.

Rule V. *If the equation be of the form*

$$x^a y^b (my\ dx + nx\ dy) = 0; \quad a, b, m, n \text{ being constants,}$$

then $x^{km-a-1} y^{kn-b-1}$, *where k has any value, is an integrating factor of the equation.*

Let us assume that $x^p y^q$ is an integrating factor of the equation

$$x^a y^b (my\ dx + nx\ dy) = 0.$$

Now $x^{p+a} y^{q+b} (my\,dx + nx\,dy)$ is an exact differential,

if $(mx^{p+a} y^{q+b+1} dx + nx^{p+a+1} y^{q+b} dy)$ be an exact differential.

This gives $m(q + b + 1) = n(p + a + 1)$

or, $\dfrac{q+b+1}{n} = \dfrac{p+a+1}{m} = k$, say, where k is any number.

Therefore $p = km - a - 1$ and $q = kn - b - 1$.

Thus we see that $x^{km-a-1} y^{kn-b-1}$, where k is any number, is an integrating factor of the equation

$$x^a y^b (my\,dx + nx\,dy) = 0.$$

In this connection, it should be observed that $\dfrac{1}{k} x^{km} y^{kn}$, $k \neq 0$ is the integral of the exact differential

$$x^{km-1} y^{kn-1} (my\,dx + nx\,dy).$$

If the equation can be put in the form

$$x^{a_1} y^{b_1} (m_1 y\,dx + n_1 x\,dy) + x^{a_2} y^{b_2} (m_2 y\,dx + n_2 x\,dy) = 0,$$

then a factor, that will make the first term an exact differential, is

$$x^{k_1 m_1 - a_1 - 1} y^{k_1 n_1 - b_1 - 1}$$

and a factor, that will make the second term an exact differential, is

$x^{k_2 m_2 - a_2 - 1} y^{k_2 n_2 - b_2 - 1}$, where k_1 and k_2 have any value.

These two factors are identical, if

$$k_1 m_1 - a_1 = k_2 m_2 - a_2 \text{ and } k_1 n_1 - b_1 = k_2 n_2 - b_2.$$

These easily determine k_1 and k_2, provided $m_1 n_2 - m_2 n_1 \neq 0$.

2.11. Illustrative Examples.

Ex. 1. *Solve* : $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$.

The given equation is

$$(hx + by + f)\,dy + (ax + hy + g)\,dx = 0$$

or, $ax\,dx + by\,dy + h(x\,dy + y\,dx) + g\,dx + f\,dy = 0$

or, $a\,d\left(\dfrac{x^2}{2}\right) + b\,d\left(\dfrac{y^2}{2}\right) + h\,d(xy) + g\,dx + f\,dy = 0$

or, $d\left\{\dfrac{1}{2}(ax^2 + by^2) + h(xy) + gx + fy\right\} = 0$.

Integrating, we get

$\frac{1}{2}(ax^2 + by^2) + hxy + gx + fy + c = 0$, where c is an arbitrary constant.

Ex. 2. *Solve* : $(1 - x^2)\dfrac{dy}{dx} - 2xy = x - x^3$.

The equation can be written as

$$(1 - x^2)\,dy - 2xy\,dx = x\,dx - x^3\,dx$$

or, $d\{(1 - x^2)y\} = d\left(\dfrac{x^2}{2} - \dfrac{x^4}{4}\right)$.

Integrating both sides, we get

$$y(1 - x^2) = \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + c.$$

Ex. 3. *Solve* $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$, *given that* $y = 1$ *when* $x = 1$.

The given equation can be put as

$$d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) + d\left(\tan^{-1}\frac{y}{x}\right) = 0.$$

Integrating, we get

$$\frac{1}{2}(x^2 + y^2) + \tan^{-1}\frac{y}{x} + c = 0, \qquad \ldots (1)$$

where c is an arbitrary constant.

Now, it is given that $y = 1$ when $x = 1$.

Putting these in (1), we get $1 + \tan^{-1} 1 + c = 0$, giving $c = -1 - \frac{1}{4}\pi$.

Therefore the required particular solution is

$$\tfrac{1}{2}(x^2 + y^2) + \tan^{-1}\frac{y}{x} = 1 + \frac{\pi}{4}.$$

Ex. 4. *Solve* $(x^2 - 2y)\,dx + (y^2 - 2x)\,dy = 0$, *given that* $y = 2$ *when* $x = 0$.

The given equation can be put as

$$x^2\,dx + y^2\,dy - 2(y\,dx + x\,dy) = 0$$

or, $d\{\tfrac{1}{3}(x^3 + y^3)\} - 2d(xy) = 0$.

Integrating, we get

$\tfrac{1}{3}(x^3 + y^3) - 2xy = c$, where c is an arbitrary constant.

Now, we have $y = 2$ when $x = 0$.

Therefore $\quad \frac{8}{3} = c$.

Hence the required particular solution is

$$x^3 + y^3 = 8 + 6xy.$$

Ex. 5. *Examine whether the equation*

$$(a^2 - 2xy - y^2)\,dx - (x + y)^2\,dy = 0$$

is exact. If it be exact, then find the primitive.

Here $M = a^2 - 2xy - y^2$ and $N = -(x + y)^2$.

We have $\frac{\partial M}{\partial y} = -2x - 2y$ and $\frac{\partial N}{\partial x} = -2(x + y)$.

Hence the equation is exact.

The primitive of the equation is

$$\int (a^2 - 2xy - y^2)\,dx + \int (-y^2)\,dy = 0,$$

y is considered as constant in the first integral

or, $a^2x - x^2y - xy^2 - \frac{1}{3}y^3 = c$.

Note. The first integral is $\int M\,dx$ and the second integral is $\int$ (terms not containing x in N) dy.

2.12. Linear equation.

An equation of the form

$$\frac{dy}{dx} + Py = Q, \qquad \ldots \quad (1)$$

where P and Q are functions of x only (or constants) is called a *linear equation* of first order in y. The dependent variable as also its derivative in such equations occur in the first degree only and not as higher powers or products.

If both P and Q be constants, then the variables can be easily separated. This will also happen if either P or Q be zero.

Let R be an integrating factor of the above equation. Then the left hand side of the equation

$$R\frac{dy}{dx} + RPy = RQ$$

is the differential coefficient of some product. Now the first term $R\frac{dy}{dx}$ can only be derived by differentiating Ry.

We put

$$R\frac{dy}{dx} + RPy = \frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}.$$

Therefore $RP = \frac{dR}{dx}$.

Integrating, we get $\log R = \int P\,dx$, so that $R = e^{\int P\,dx}$ is an integrating factor.

Now, multiplying both sides of the equation (1) by this integrating factor, we get

$$\frac{dy}{dx}e^{\int P\,dx} + Pye^{\int P\,dx} = Qe^{\int P\,dx}$$

or, $d\left(ye^{\int P\,dx}\right) = Qe^{\int P\,dx}\,dx$.

Integrating, we get the primitive as

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c,$$ where c is a constant.

An alternative form of the solution of the equation (1).

Let $z = \dfrac{Q}{P} - y$, that is, $Pz = Q - Py$. ... (2)

Then $\dfrac{dz}{dx} = \dfrac{d}{dx}\left(\dfrac{Q}{P}\right) - \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{Q}{P}\right) - (Q - Py)$, from (1)

$= \dfrac{d}{dx}\left(\dfrac{Q}{P}\right) - Pz$, from (2).

Therefore $\dfrac{dz}{dx} + Pz = \dfrac{d}{dx}\left(\dfrac{Q}{P}\right)$, which is linear in z.

The integrating factor of this equation is $e^{\int P\,dx}$. Multiplying the equation by the integrating factor $e^{\int P\,dx}$, we get $\dfrac{d}{dx}\left(ze^{\int P\,dx}\right) = e^{\int P\,dx}\dfrac{d}{dx}\left(\dfrac{Q}{P}\right)$.

On integration, we get $ze^{\int P\,dx} = \int e^{\int P\,dx}\dfrac{d}{dx}\left(\dfrac{Q}{P}\right)dx + c$.

Putting for z, we get an alternative form for the solution of the equation (1) as $\dfrac{Q}{P} - y = e^{-\int P\,dx}\left[\int e^{\int P\,dx}\,d\left(\dfrac{Q}{P}\right) + c\right]$

or, $y = \dfrac{Q}{P} - e^{-\int P\,dx}\left[\int e^{\int P\,dx}\,d\left(\dfrac{Q}{P}\right) + c\right]$.

Note. Sometimes an equation may be linear in x, where y is the independent variable. The form of such an equation is

$$\frac{dx}{dy} + P_1 x = Q_1 .$$

Here P_1 and Q_1 are functions of y only (or constants).

The general solution, in this case, will be

$$xe^{\int P_1\,dy} = \int Q_1 e^{\int P_1\,dy}\,dy + c.$$

2.13. Equation reducible to linear form.

Let us consider the equation $\dfrac{dy}{dx} + Py = Qy^n$, which is known as *Bernoulli's equation*, in which P and Q are functions of x alone or constants.

It can be put as $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$.

This equation may be brought to the linear form by the substitution

$y^{1-n} = v$, so that $\dfrac{dv}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$.

Thus the equation transforms to

$$\frac{dv}{dx} + (1 - n) Pv = (1 - n) Q,$$

which is linear in v, its integrating factor being $e^{(1-n)\int P\,dx}$

The solution is given by

$$ve^{(1-n)\int P\,dx} = (1-n)\int Qe^{(1-n)\int P\,dx}\,dx + c.$$

Then we put y^{1-n} for v.

2.14. Method of variation of parameters.

Consider the equation $\frac{dy}{dx} + Py = Q$, ... (1)

where P and Q are functions of x not involving y.

If the function Q be zero, then the integral of the equation can easily be found and will be found to be

$$y = ce^{-\int P\,dx},$$

where c is a constant.

When Q is different form zero, we assume the same form for y, but do not restrict c to be constant and let it be u, a function of x, so that $y = u\,e^{-\int P\,dx}$. ... (2)

From (2), we get

$$\frac{dy}{dx} = \left(\frac{du}{dx} - Pu\right) e^{-\int P\,dx}$$

and hence from (2) and (1), $\frac{du}{dx} e^{-\int P\,dx} = \frac{dy}{dx} + P\,u\,e^{-\int P\,dx} = Q$,

so that $\frac{du}{dx} = Q\,e^{\int P\,dx}$,

which gives $u = \int Q\,e^{\int P\,dx}\,dx + C$, ... (3)

where C is an arbitrary constant.

Thus putting (3) in (2), we get

$$y = Ce^{-\int P\,dx} + e^{-\int P\,dx}\int Q\,e^{\int P\,dx}\,dx. \quad \ldots (4)$$

(4) is the primitive of the linear differential equation (1).

Note 1. This method of obtaining an integral of the equation when Q is zero, and then treating the parameter c in the integral as actually variable, when Q is not zero, is called the *method of variation of parameters*. This method is frequently used in the solution of linear equations of second and higher orders and will be discussed later.

Note 2. Bernoulli's equations can also be solved by the method of variation of parameters after reducing it to linear form.

2.15. Illustrative Examples.

Ex. 1. *Solve* : $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1}y = \dfrac{1}{(x^2+1)^3}$.

The equation is in linear form in y.

Here $P = \dfrac{4x}{x^2+1}$ and $Q = \dfrac{1}{(x^2+1)^3}$.

Integrating factor is $e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = (x^2+1)^2$.

Multiplying both sides of the equation by this integrating factor, we get

$$(x^2+1)^2\frac{dy}{dx} + 4x(x^2+1)y = \frac{1}{x^2+1}$$

or, $$d\left\{y(x^2+1)^2\right\} = \frac{1}{x^2+1}dx.$$

Integrating both sides, we get

$$y(x^2+1)^2 = \tan^{-1}x + c, \text{ where } c \text{ is a constant.}$$

Ex. 2. *Solve* : $1 + y^2 + (x - e^{-\tan^{-1}y})\dfrac{dy}{dx} = 0$.

The equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2},$$

which is linear in x.

Here $P = \dfrac{1}{1+y^2}$ and $Q = \dfrac{e^{-\tan^{-1}y}}{1+y^2}$.

Integrating factor is

$$e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Multiplying both sides of the equation by this integrating factor, we get

$$\frac{dx}{dy} e^{\tan^{-1} y} + \frac{e^{\tan^{-1} y}}{1+y^2} x = \frac{1}{1+y^2}$$

or, $\quad d\left(x e^{\tan^{-1} y}\right) = \frac{1}{1+y^2} dy .$

Integrating both sides, we get the general solution as

$$x e^{\tan^{-1} y} = \tan^{-1} y + c .$$

Ex. 3: *Solve* : $\quad \frac{dy}{dx} + y \cos x = y^n \sin 2x .$

We divide both sides of the equation by y^n and get

$$y^{-n} \frac{dy}{dx} + y^{1-n} \cos x = \sin 2x .$$

Put $y^{1-n} = v$, so that $(1-n)\, y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$ and the equation becomes

$$\frac{1}{1-n} \frac{dv}{dx} + v \cos x = \sin 2x$$

or, $\quad \frac{dv}{dx} + (1-n) v \cos x = (1-n) \sin 2x .$

Integrating factor of this linear equation in v is

$$e^{\int (1-n) \cos x \, dx} = e^{(1-n) \sin x}$$

The solution of this equation is thus

$$v e^{(1-n) \sin x} = \int (1-n) e^{(1-n) \sin x} \sin 2x \, dx . \qquad \dots (1)$$

Now, to evaluate the right hand side integral, we put $\sin x = z$, so that $\cos x \, dx = dz$.

Therefore $(1-n) \int 2e^{(1-n)\sin x} \sin x \cos x \, dx = 2(1-n) \int e^{(1-n)z} z \, dz$

$$= 2(1-n) \left\{ z \frac{1}{1-n} e^{(1-n)z} - \int \frac{1}{1-n} e^{(1-n)z} dz \right\}$$

$$= 2ze^{(1-n)z} - 2 \cdot \frac{1}{1-n} e^{(1-n)z} + c$$

$$= 2 \sin x \, e^{(1-n) \sin x} - \frac{2}{1-n} e^{(1-n) \sin x} + c .$$

Putting $v = y^{1-n}$ in (1), we get the general solution of the given equation as

$$y^{1-n} e^{(1-n) \sin x} = 2 \sin x \, e^{(1-n) \sin x} - \frac{2}{1-n} e^{(1-n) \sin x} + c$$

or, $\quad y^{1-n} = 2 \sin x - \frac{2}{1-n} + c e^{(n-1) \sin x} .$

CHAPTER III

EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

3.1. Introduction.

In this chapter, we consider those differential equations, which involve $\frac{dy}{dx}$ in a higher degree than one. For brevity, $\frac{dy}{dx}$ is usually denoted by the symbol p. Let us consider an equation of the first order and n-th degree in p of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots\cdots\cdots + P_{n-1} p + P_n = 0,$$

where $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y.

We shall discuss three special cases of the above equation, in which it is

(*i*) solvable for p, (*ii*) solvable for y, (*iii*) solvable for x.

3.2. Equations solvable for p.

Let us suppose first that the left hand side of the above equation can be expressed as a product of n linear factors in p, that is, the above equation can be put in the form

$$\{p - f_1(x, y)\}\{p - f_2(x, y)\}\ldots\ldots\{p - f_n(x, y)\} = 0.$$

This is true if one or more of the factors on the left hand side be equal to zero. Thus any relation connecting x and y, which will make one or more of the factors zero, will be a solution of the equation. This is true for any factor. Thus each factor equated to zero will give a solution to the above equation. Let the solutions be

$$F_1(x, y, c_1) = 0, F_2(x, y, c_2) = 0, \ldots, F_n(x, y, c_n) = 0, \quad \ldots (1)$$

where $c_1, c_2, \ldots\ldots, c_n$ are constants.

All possible solutions of the above equation will be then included in the relation

$$F_1(x, y, c_1)\, F_2(x, y, c_2) \ldots\ldots F_n(x, y, c_n) = 0. \quad \ldots (2)$$

Now, as the given equation is only of the first order, we expect only one arbitrary constant in its general solution. We also observe that, there is no loss of generality, if the arbitrary constants $c_1, c_2, \ldots\ldots, c_n$ be replaced by a single arbitrary constant c, because every particular solution which is obtained from (1), can also be obtained from (2), (if $c_1, c_2, \ldots\ldots, c_n$ be all replaced by c) by giving a suitable value to c.

Thus the general complete primitive of the above equation is

$$F_1(x, y, c)\, F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0,$$

where c is any arbitrary constant.

Note. The degree of p in the given equation is the same as that of c in its general complete primitive.

3.3. Illustrative Examples.

Ex. 1. *Solve* : $p^2 + 2xp - 3x^2 = 0$. [*C. H. 1987*]

The equation is a quadratic in p.

This equation can be written in the factorised form as

$$(p - x)(p + 3x) = 0.$$

The first factor, when equated to zero, gives $p - x = 0$, that is, $\frac{dy}{dx} = x$, whose solution is

$$2y - x^2 + c_1 = 0$$

The second factor, when equated to zero, gives $p + 3x = 0$, that is, $\frac{dy}{dx} = -3x$, whose solution is

$$2y + 3x^2 + c_2 = 0.$$

Hence the general solution of the above equation is

$(2y - x^2 + c)(2y + 3x^2 + c) = 0$, where c is an arbitrary constant.

Ex. 2. *Solve* :

$$p^3 - (x^2 + xy + y^2)\, p^2 + (x^3 y + x^2 y^2 + xy^3)\, p - x^3 y^3 = 0.$$

[*V. H. 1992*]

On factorization, the given equation becomes

$$(p - x^2)(p - xy)(p - y^2) = 0.$$

The first factor, when equated to zero, gives

$p - x^2 = 0$, which has $x^3 - 3y + c_1 = 0$ as its solution.

The second factor, when equated to zero, gives

$p - xy = 0$, which has $e^{\frac{1}{2}x^2} + c_2 y = 0$ as its solution.

The third factor, when equated to zero, gives

$p - y^2 = 0$, which has $xy + yc_3 + 1 = 0$ as its solution.

3.4. Equations solvable for y.

If the differential equation be solvable for y, then it may be put in the form

$$y = f(x, p). \qquad \ldots \qquad (1)$$

Differentiating both sides of (1) with respect to x, we get an equation of the form

$$p = F\left(x, p, \frac{dp}{dx}\right).$$

This is an equation in two variables x and p and it can be solved to get a solution of the form

$$\phi(x, p, c) = 0. \qquad \ldots \qquad (2)$$

Eliminating p between (1) and (2), we shall get the required solution, which will be a relation connecting x, y and an arbitrary constant c.

Sometimes we write down the solution by expressing x and y separately as functions of p, treating p as a parameter. This happens when the elimination of p between (1) and (2) cannot be easily done.

3.5. Equations containing no x.

Consider the equation of the form $f(y, p) = 0$.

If it be solvable for p, then it can be put as

$$p = \frac{dy}{dx} = F(y).$$

Its solution is
$$\int \frac{dy}{F(y)} = x + c.$$

On the other hand, if it be solvable for y, let it be
$$y = f(p).$$

This can be easily integrated by the previous method.

3.6. Equations solvable for x.

If the differential equation be solvable for x, then it may be put in the form
$$x = f(y, p). \qquad \ldots (1)$$

Differentiating both sides of (1) with respect to y, we get an equation of the form
$$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right).$$

This is an equation in two variables y and p and it can be solved to get a solution of the form
$$\phi(y, p, c) = 0. \qquad (2)$$

Eliminating p between (1) and (2), we shall get the required solution, which will be a relation connecting x, y and an arbitrary constant c.

In case the elimination is not easily possible, x and y may be expressed in terms of p, treating p as a parameter.

3.7. Equations containing no y.

Consider the equation of the form $f(x, p) = 0$.

If it be solvable for p, then it can be put as
$$p = \frac{dy}{dx} = F(x)$$

and its solution is $\int F(x)\,dx = y + c$.

On the other hand, if it be solvable for x, let it be
$$x = f(p).$$

This can be easily integrated by the previous method.

3.8. Illustrative Examples.

Ex. 1. *Solve* : $y + px = x^4p^2$.

The equation is solvable for y and is written as

$$y = -px + x^4p^2. \quad \ldots (1)$$

Differentiating both sides of the equation with respect to x and writing p for $\frac{dy}{dx}$, we have

$$p = -p - x\frac{dp}{dx} + x^4.2p\frac{dp}{dx} + p^2.4x^3$$

or, $$2p + x\frac{dp}{dx} = 2px^3\left(2p + x\frac{dp}{dx}\right)$$

or, $$\left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0.$$

Therefore $2p + x\frac{dp}{dx} = 0$, if $1 - 2px^3 \neq 0$.

This gives $$\frac{2}{x}dx + \frac{dp}{p} = 0.$$

Integrating, we get

$$\log x^2 + \log p = \log c$$

or, $px^2 = c$, where c is an arbitrary constant

or, $$p = \frac{c}{x^2}. \quad \ldots (2)$$

Eliminating p between (1) and (2), we get the general solution as

$$y = -x\frac{c}{x^2} + x^4 \cdot \frac{c^2}{x^4}$$

or, $$xy = c^2x - c. \quad \ldots (3)$$

In case $1 - 2px^3 = 0$, we have $p = \frac{1}{2}x^{-3}$ and putting this value of p in the given equation, we get

$$y = -\tfrac{1}{2}x^{-2} + \left(\tfrac{1}{2}x^{-3}\right)^2 \cdot x^4 = -\tfrac{1}{4}x^{-2}$$

or, $4x^2y + 1 = 0$.

Although this relation satisfies the given differential equation, this does not contain an arbitrary constant nor it can be deduced from the general solution by giving a particular value to the arbitrary constant c in (3). This is called the *singular solution* of the given equation.

3.9. Clairaut's equation.

A differential equation of the form

$$y = px + f(p)$$

is known as *Clairaut's equation.*

To solve this equation, we differentiate its both sides with respect to x and obtain

$$p = p + \{x + f'(p)\}\frac{dp}{dx},$$

from which we get *either* $\frac{dp}{dx} = 0$, that is, $p = c$, a constant

or $\qquad x + f'(p) = 0$, a relation between x and p.

Eliminating p from the given equation and $p = c$, that is, putting $p = c$ in the given equation, we get the complete primitive as

$$y = cx + f(c).$$

This is the *general solution* of the given differential equation.

Eliminating p from the given equation and $x + f'(p) = 0$, we get a relation between x and y.

On differentiating the complete primitive with respect to x, we get

$$p = c$$

and eliminating c from $y = cx + f(c)$,

we get the corresponding differential equation

$$y = px + f(p).$$

Evidently, $\qquad y = cx + f(c)$

is a solution of the equation.

If now we look to the other relation between x and y which is obtained by eliminating p between

$$y = px + f(p) \text{ and } x + f'(p) = 0,$$

we see at once that it contains no arbitrary constant and it cannot be deduced from the general solution by giving any particular value to the arbitrary constant. Yet it may be seen to be a solution of the equation ; for, differentiating the first equation with respect to x, we get

$$\frac{dy}{dx} = p + \{x + f'(p)\} \frac{dp}{dx}$$

$$= p, \text{ since } x + f'(p) = 0 \text{ and } \frac{dp}{dx} \text{ is finite.}$$

The second solution of the equation, which is derived from the equation in the above manner and which is not included in the general solution, is called the *singular solution* of the equation.

Note 1. The relation between the two solutions, if both exist, can be indicated by some geometrical consideration. The first solution

$$y = cx + f(c) \qquad \ldots \quad (1)$$

represents a family of straight lines. If this family of straight lines has an envelope, then it can be found by differentiating both sides of the equation (1) with respect to c, which gives

$$0 = x + f'(c), \qquad \ldots \quad (2)$$

and then eliminating c between (1) and (2).

Now, elimination of c between (1) and (2) is precisely the same as that of p between

$$y = px + f(p) \text{ and } 0 = x + f'(p).$$

Thus the curve given by the latter is the envelope of the family of straight lines represented by the general solution, if these lines have an envelope.

Note 2. It can be easily shown that the solutions obtained above are distinct ; for, let

$$U = y - cx - f(c) \text{ and } V = y - px - f(p),$$

where, in V, the value of p is given by $x + f'(p) = 0$.

Here $\quad \frac{\partial U}{\partial x} = -c, \frac{\partial U}{\partial y} = 1, \frac{\partial V}{\partial y} = 1 \text{ and } \frac{\partial V}{\partial x} = -p.$

Now $\quad \frac{\partial U}{\partial x} \cdot \frac{\partial V}{\partial y} - \frac{\partial U}{\partial y} \cdot \frac{\partial V}{\partial x} = p - c,$

which does not vanish identically. It vanishes only when the equations $U = 0$ and $V = 0$ are taken simultaneously. Hence the two equations $U = 0$ and $V = 0$ are independent of one another.

The solutions are therefore distinct from one another.

3.10. Equations homogeneous in x and y.

This type of equation can be put in the form

$$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0, \text{ that is, } F\left(p, \frac{y}{x}\right) = 0.$$

The case, in which it is solvable for p, has already been discussed.

If it be solvable for $\frac{y}{x}$, then it can be put in the form

$$\frac{y}{x} = f(p), \text{ that is, } y = x\, f(p).$$

Differentiating both sides with respect to x, we get

$$p = f(p) + x\, f'(p) \frac{dp}{dx}$$

or,
$$\frac{dx}{x} = \frac{f'(p)\, dp}{p - f(p)},$$

in which variables have been separated and can be easily integrated.

Then the p-eliminant of this integral and the given equation will be the complete primitive.

3.11. Lagrange's equation.

Consider an equation of the form

$$y = x\, f(p) + \phi(p),$$

which is an extended form of Clairaut's equation.

Differentiating its both sides with respect to x, we get

$$p = f(p) + \{x\, f'(p) + \phi'(p)\} \frac{dp}{dx}$$

or,
$$p - f(p) = \{x\, f'(p) + \phi'(p)\} \frac{dp}{dx}$$

or,
$$\frac{dx}{dp} + \frac{x f'(p)}{f(p) - p} = \frac{\phi'(p)}{p - f(p)},$$

which is a linear equation of the first order in x. This can be easily integrated and the p-eliminant of this integral and the given equation will be the complete primitive.

Note. Sometimes a change of variable is to be applied to a given equation to reduce the equation to Clairaut's form or to its extended form. This will be illustrated through examples.

3.12. Illustrative Examples.

Ex. 1. *Obtain the complete primitive and the singular solution of the equation*

$$y = px + \sqrt{1+p^2}.$$ [C. H. 1987]

The equation is in Clairaut's form.

Differentiating both sides of this equation with respect to x, we get

$$p = p + x\frac{dp}{dx} + \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

or, $$\left(x + \frac{p}{\sqrt{1+p^2}}\right)\frac{dp}{dx} = 0.$$

Therefore, *either* $\frac{dp}{dx} = 0$, that is, $p = c$, a constant ... (1)

or $x + \frac{p}{\sqrt{1+p^2}} = 0$, that is, $x = -\frac{p}{\sqrt{1+p^2}}$. ... (2)

Eliminating p between the given equation and (1), we get the complete solution as

$$y = cx + \sqrt{1+c^2}.$$

Eliminating p between the given equation and (2), we get the singular solution as

$$x^2 + y^2 = \frac{p^2}{1+p^2} + \left(\frac{-p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2}\right)^2 = 1.$$

Note. Notice that the complete primitive represents a family of straight lines, which touch the circle given by the singular solution at different points for different values of the arbitrary constant.

Ex. 2. *Reduce the differential equation* $y = 2px - p^2y$ *to Clairaut's form by the substitutions* $y^2 = Y$, $x = X$ *and then obtain the complete primitive and singular solution, if any.* [C. H. 1986]

We substitute $y^2 = Y$ and $x = X$, so that we have $2py = \frac{dY}{dx}$ and $dx = dX$, in which we write $p = \frac{dy}{dx}$. Therefore $\frac{dY}{dX} = 2py$.

Let $$P = \frac{dY}{dX}. \text{ Hence } \frac{P}{2y} = p.$$

Thus the given equation reduces to

$$y = 2x\cdot\frac{P}{2y} - \frac{P^2}{4y^2}\cdot y$$

or, $$Y = PX - \tfrac{1}{4}P^2.$$ ... (1)

This equation is in Clairaut's form.

Differentiating both sides of (1) with respect to X, we get

$$P = P + \left(X - \tfrac{1}{2}P\right)\frac{dP}{dX}$$

or, $$\frac{dP}{dX}\left(X - \tfrac{1}{2}P\right) = 0.$$

Therefore, *either* $\dfrac{dP}{dX} = 0$, which gives $P = C$, a constant ... (2)

or $X - \tfrac{1}{2}P = 0$, which gives $P = 2X$. ... (3)

Eliminating P between (1) and (2), we get the complete primitive as

$$Y = CX - \tfrac{1}{4}C^2.$$

Restoring the values of X and Y, we get the primitive as

$$y^2 = Cx - \tfrac{1}{4}C^2.$$

Eliminating P between (1) and (3), we get the singular solution as

$$Y = 2X^2 - X^2 = X^2.$$

Restoring the values of X and Y, we get the singular solution as

$$y^2 = x^2, \text{ that is, } y = \pm x.$$

Ex. 3. *Solve :* $x^2(y - px) = p^2y$. [*B. H. 1990*]

Let us put $x^2 = u$ and $y^2 = v$,

so that $2x\,dx = du$ and $2y\,dy = dv$.

Therefore $$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$$

or, $$\frac{y}{x}p = \frac{dv}{du} = P \text{ (say)}.$$

Putting for P in the given equation, we get

$$x^2\left(y - \frac{Px^2}{y}\right) = \frac{x^2}{y^2}P^2 . y$$

or, $$y^2 - Px^2 = P^2$$

or, $$v - Pu = P^2$$

or, $v = Pu + P^2$, which is in Clairaut's form.

The complete primitive is thus

$v = cu + c^2$, c being an arbitrary constant.

Restoring the values of u and v, we get

$$y^2 = cx^2 + c^2.$$

CHAPTER IV

SINGULAR SOLUTIONS

4.1. Definitions.

Sometimes a solution of a differential equation can be found without involving any arbitrary constant and which is, in general, not a particular case of the general solution. Such a solution we got when we solved equations of Clairaut's form and we named it a *singular solution.* Now we shall attempt a general discussion on such solutions.

The *discriminant* of an equation involving a single variable is the simplest function of the coefficients in a rational integral form. The vanishing of this discriminant is the condition that the equation has multiple roots. Thus the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $(b^2 - 4ac)$ and $b^2 - 4ac = 0$ will be called the *discriminant relation.* In the case of a general equation, such as $F(x, y, \alpha) = 0$, in which α is a variable and the coefficients are functions of x and y, we obtain the discriminant by eliminating α between $F = 0$ and $\frac{\partial F}{\partial \alpha} = 0$. This eliminant is the simplest rational function of x and y whose vanishing ensures that the equation has equal roots for α and is called the α-*discriminant* relation or simply α-*discriminant* of F.

Let us consider the differential equation

$$f(x, y, p) = 0,$$

whose general solution is $\phi(x, y, c) = 0$, c being an arbitrary constant.

The c-discriminant is obtained by eliminating c between

$$\phi(x, y, c) = 0 \text{ and } \frac{\partial \phi}{\partial c} = 0. \quad \ldots (1)$$

The p-discriminant is obtained by eliminating p between the equations

$$f(x, y, p) = 0 \text{ and } \frac{\partial f}{\partial p} = 0. \quad \ldots (2)$$

Evidently, the c-discriminant is the locus of the points for each of which $\phi(x, y, c) = 0$ has equal values of c and the p-discriminant is the locus of the points for each of which $f(x, y, p) = 0$ has equal values of p. The equation $f(x, y, p) = 0$ and its solution $\phi(x, y, c) = 0$ are of the same degree in p and c respectively and hence if there be a p-discriminant, then there must be a c-discriminant.

4.2. Envelope.

By giving c all possible values in $\phi(x, y, c) = 0$, we obtain a set of curves, infinite in number, of the same kind. Let these successive values of c differ by infinitesimal amounts. When these curves are drawn, the curves corresponding to two consecutive values of c (called *consecutive curves*) intersect and the limiting position of these points of intersection includes the envelope of the system of curves. The *envelope* of the system is the locus of points of intersection of the consecutive curves of the system obtained by giving different values to c in $\phi(x, y, c) = 0$ and is obtained by eliminating c between

$$\phi(x, y, c) = 0 \text{ and } \frac{\partial \phi}{\partial c} = 0.$$

Envelope is thus a part of the locus of the c-discriminant relation, since, in the limit, the c's of two consecutive curves become equal.

Again, the envelope is touched at any point on it by some of the curves of the system. Therefore $x, y, p\left(= \frac{dy}{dx}\right)$ for any point on the envelope are identical with $x, y, p\left(= \frac{dy}{dx}\right)$ of some point on one of the curves of the system. At the points of ultimate intersection of consecutive curves, the p's for the intersecting curves become equal. Thus the locus of the points where the p's have equal values, will include the envelope. The p-discriminant relation of $f(x, y, p) = 0$ contains the equation of the envelope of the system of curves given by $\phi(x, y, c) = 0$ and is also a solution of the differential equation $f(x, y, p) = 0$. Thus both the p-discriminant and c-discriminant relations contain the equation to the envelope. This is the *singular solution* and this satisfies the differential equation. This does not contain any arbitrary constant and cannot be deduced from the complete primitive by giving a particular value to the arbitrary constant, except in some special cases.

As an example, let us consider the differential equation

$$4xp^2 = (3x - a)^2.$$

This gives $\quad \dfrac{dy}{dx} = \dfrac{3x - a}{2\sqrt{x}}$

or, $\quad dy = \left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right) dx.$

Integrating, we get the general solution as

$$y+c=x^{\frac{3}{2}}-ax^{\frac{1}{2}}, \text{ where } c \text{ is an arbitrary constant}$$

or, $$(y+c)^2=x(x-a)^2. \qquad \cdots(1)$$

Now, let $\phi(x,y,c)=(y+c)^2-x(x-a)^2=0.$

Therefore $\dfrac{\partial\phi}{\partial c}=2(y+c)=0.$

Eliminating c, we get the c-discriminant as $x(x-a)^2=0. \qquad \cdots(2)$

Again, from the given equation, we have

$$f(x,y,p)=4xp^2-(3x-a)^2=0,$$

which is a quadratic equation in p. Hence the p-discriminant is

$$x(3x-a)^2=0. \qquad \cdots(3)$$

From (2) and (3), we have the common factor x for the c-discriminant and the p-discriminant and hence $x=0$ is the singular solution. This satisfies the given equation and does not contain any arbitrary constant. Notice further that the singular solution $x=0$ cannot be found from the general solution (1) by giving any particular value to the arbitrary constant.

Observe that $(x-a)^2=0$ of the c-discriminant and $(3x-a)^2=0$ of the p-discriminant do not satisfy the differential equation and are thus not the solutions. Hence we can say that the c-discriminant and the p-discriminant will contain other loci also, which are not the solutions.

Note that the straight line $x=0$ is touched by the family of curves represented by the general solution and hence is the envelope of the family.

It can be easily verified from above that the c-discriminant is the locus of each point for which $\phi(x,y,c)=0$ has equal values of c and p-discriminant is the locus of each point for which $f(x,y,p)=0$ has equal values of p.

Thus, this example establishes that, while p- and c-discriminant relations must both contain the singular solution, which represents the envelope, if there be one, they may contain some other loci, which are not the solutions of the differential equation. In the next article, we shall discuss the nature of such loci, which may occur in the discriminants.

4.4. Illustrative Examples.

Ex. 1. *Examine the equation* $y = 2px + p^2$ *for singular solution.*

[C. H. 1984, 1997]

We have $$(p + x)^2 = y + x^2$$

or, $$p = \sqrt{y + x^2} - x.$$

Put $y = vx^2$, so that $p = x^2 \dfrac{dv}{dx} + 2xv = x(\sqrt{1+v} - 1)$

or, $$x\frac{dv}{dx} = \sqrt{1+v} - 1 - 2v$$

or, $$\frac{dx}{x} = \frac{dv}{\sqrt{1+v} - 1 - 2v}.$$

Putting $u^2 = 1 + v$, we have

$$\frac{dx}{x} = \frac{2u\,du}{1 + u - 2u^2} = -\frac{1}{2}\frac{(1-4u)\,du - du}{1 + u - 2u^2}.$$

Integrating, we get

$$\log x = -\tfrac{1}{2}\log(1 + u - 2u^2) + \tfrac{1}{2}\int \frac{du}{1 + u - 2u^2}$$

$$= -\tfrac{1}{2}\log(1 + u - 2u^2) - \tfrac{1}{6}\log\frac{1-u}{1+2u} + \text{a constant.}$$

Hence $x^6(1 + u - 2u^2)^3 \dfrac{1-u}{1+2u} = \text{a constant}$

or, $x^6(1-u)^4(1+2u)^2 = \text{a constant}$

or, $x^3(1-u)^2(1+2u) = \text{another constant} = c$ (say)

or, $1 - 3u^2 + 2u^3 = \dfrac{c}{x^3}$

or, $1 - 3\left(1 + \dfrac{y}{x^2}\right) + 2\left(1 + \dfrac{y}{x^2}\right)^{\frac{3}{2}} = \dfrac{c}{x^3}$

giving finally the general solution as

$$(2x^3 + 3xy + c)^2 = 4(x^2 + y)^3.$$

The c-discriminant is obtained from the condition that the two values of c of the general solution are equal. This gives the c-discriminant as $(x^2 + y)^3 = 0$, in which $x^2 + y = 0$ occurs thrice. Again, the p-discriminant is obtained from the condition that the two values of p are equal in the given equation. This gives the p-discriminant as $x^2 + y = 0$, which occurs once only.

We notice that this does not satisfy the given equation.

Hence $x^2 + y = 0$ is the cusp-locus and the equation has no singular solution.

Ex. 2. *Find the singular solution of the differential equation*

$$\sin\left(x\frac{dy}{dx}\right)\cos y = \cos\left(x\frac{dy}{dx}\right)\sin y + \frac{dy}{dx}.$$ [C. H. 1994]

The equation can be put as

$$\sin\left(x\frac{dy}{dx} - y\right) = \frac{dy}{dx}$$

or, $$xp - y = \sin^{-1}p$$

or, $$y = px - \sin^{-1}p, \quad \ldots \quad (1)$$

which is in Clairaut's form and its complete primitive is

$$y = cx - \sin^{-1}c. \quad \ldots \quad (2)$$

Now, differentiating both sides of (1) partially with respect to p, we get

$$0 = x - \frac{1}{\sqrt{1-p^2}}, \text{ which gives } \quad p = \frac{\sqrt{x^2-1}}{x}. \quad \ldots \quad (3)$$

Eliminating p between (1) and (3), we get the p-discriminant relation as

$$y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}. \quad \ldots \quad (4)$$

c-discriminant obviously will be the same.

Hence we have the singular solution of he given equation as

$$y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}.$$

Ex. 3. *Solve and examine for singular solution of the equation*

$$xp^2 - (x-a)^2 = 0.$$

The given equation can be written as

$$p = \pm\frac{x-a}{\sqrt{x}} = \pm\left(\sqrt{x} - ax^{-\frac{1}{2}}\right)$$

or, $$dy = \pm\left(\sqrt{x} - ax^{-\frac{1}{2}}\right)dx.$$

Integrating both sides, we get

$$y = c \pm \left(\tfrac{2}{3}x^{\frac{3}{2}} - 2ax^{\frac{1}{2}}\right)$$

$$= c \pm 2\sqrt{x}\left(\tfrac{1}{3}x - a\right)$$

$$= c \pm \tfrac{2}{3}\sqrt{x}\,(x-3a).$$

The complete primitive is thus

$$(y-c)^2 = \tfrac{4}{9}x\,(x-3a)^2.$$

From the given differential equation, we get the p-discriminant relation as

$$x(x-a)^2 = 0. \qquad \dots (1)$$

From the complete primitive, the c-discriminant relation is obtained as

$$x(x-3a)^2 = 0. \qquad \dots (2)$$

From (1) and (2), it is seen that x is the common factor in both the discriminants. Hence $x = 0$ is the singular solution. Furthermore $x - a = 0$ occurs twice in the p-discriminant; hence this is the tac-locus. Also $x - 3a = 0$ occurs twice in the c-discriminant; hence $x - 3a = 0$ gives the nodal locus.

Ex. 4. *Find the singular solution of the differential equation satisfied by the family of curves $c^2 + 2cy - x^2 + 1 = 0$, where c is a parameter.*

[C. H. 1987]

The equation of the family of curves is

$$c^2 + 2cy - x^2 + 1 = 0. \qquad \dots (1)$$

Differentiating both sides of (1) with respect to x, we get

$$2c\frac{dy}{dx} - 2x = 0, \text{ giving } c = \frac{x}{p}.$$

Putting for c in (1), we get the corresponding differential equation

$$\frac{x^2}{p^2} + 2\frac{x}{p}y - x^2 + 1 = 0$$

or, $$x^2 + 2xyp + (1 - x^2)p^2 = 0. \qquad \dots (2)$$

From (1), the c-discriminant relation is

$$4y^2 - 4(1 - x^2) = 0, \text{ that is, } x^2 + y^2 - 1 = 0.$$

From (2), the p-discriminant relation is

$$4x^2y^2 - 4x^2(1 - x^2) = 0$$

or, $$x^2(x^2 + y^2 - 1) = 0.$$

Hence the singular solution is

$$x^2 + y^2 - 1 = 0, \text{ that is, } x^2 + y^2 = 1.$$

$x = 0$ occurs twice in the p-discriminant. Hence $x = 0$ gives the tac-locus.

Ex. 5. *Solve and test for singular solution of the equation*

$$p^3 - 4xyp + 8y^2 = 0.$$

Substituting $y = v^2$, we have $p = \dfrac{dy}{dx} = 2v\dfrac{dv}{dx}$.

Then the given equation becomes

$$8v^3\left(\frac{dv}{dx}\right)^3 - 4x \cdot v^2 \cdot 2v\frac{dv}{dx} + 8v^4 = 0$$

or,

$$\left(\frac{dv}{dx}\right)^3 - x\frac{dv}{dx} + v = 0$$

or,

$$v = x\frac{dv}{dx} - \left(\frac{dv}{dx}\right)^3,$$

which is in Clairaut's form and its complete primitive is

$$v = c_1 x - c_1^3$$

or,

$$\sqrt{y} = c_1 x - c_1^3$$

or,

$$y = c(x - c)^2, \text{ putting } c_1^2 = c. \qquad \ldots \quad (1)$$

Now, differentiating both sides of the given equation partially with respect to p, we get

$$3p^2 - 4xy = 0.$$

Eliminating p between this and the given equation, we get

$$y^3(27y - 4x^3) = 0, \qquad \ldots \quad (2)$$

which is thus the p-discriminant relation.

Now, differentiating both sides of the complete primitive (1) with respect to c, the parameter, we get

$$-2c(x - c) + (x - c)^2 = 0$$

or,

$$(x - c)(x - 3c) = 0. \qquad \ldots \quad (3)$$

Eliminating c between (1) and (3), we get

$$y = 0 \text{ (when } c = x) \text{ and } 27y - 4x^3 = 0 \quad (\text{when } c = \tfrac{1}{3}x).$$

Therefore the c-discriminant relation is

$$y(27y - 4x^3) = 0. \qquad \ldots \quad (4)$$

From (2) and (4), we can say that $y = 0$ and $27y - 4x^3 = 0$ are the singular solutions.

Note. Notice that $y = 0$ is a particular solution of the equation corresponding to $c = 0$.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

5.1. Complementary function and particular integral.

An ordinary linear differential equation of n-th order has the form

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = X, \quad \ldots \quad \text{(I)}$$

where X (called the *input function*) and the coefficients $P_1, P_2, \ldots\ldots, P_n$ are constants or functions of x only. The dependent variable and its derivatives appear only in the first degree and are not multiplied together. If the coefficient of the derivative of the highest order $\frac{d^ny}{dx^n}$ be not unity, then all the terms of the equation can be divided by that coefficient, so that (I) is the most general form of such equations.

In this chapter, we shall consider only the ordinary linear differential equations, in which $P_1, P_2, \ldots\ldots, P_n$ are constants and X is a function of x only or a constant. We shall consider two forms of the equation (I). First we consider the form, in which the right hand member, that is, X is zero and then we consider the form, in which X is a function of x only or a constant. Equations of the first form (that is, when $X = 0$) are said to be *homogeneous.*

Theorem 1. *If $y = f(x)$ be the general solution of the equation*

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = 0 \quad \ldots \quad (1)$$

and $y = \phi(x)$ be a solution of the equation

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = X, \quad \ldots \quad (2)$$

then

$$y = f(x) + \phi(x) \quad \ldots \quad (3)$$

is the general solution of the equation (2).

Substituting the value of y from (3) in (2), we get the left hand side of (2) equal to

$$\left(\frac{d^nf}{dx^n} + P_1\frac{d^{n-1}f}{dx^{n-1}} + \cdots\cdots + P_n f\right) + \left(\frac{d^n\phi}{dx^n} + P_1\frac{d^{n-1}\phi}{dx^{n-1}} + \cdots\cdots + P_n\phi\right).$$

Now, $y = f(x)$ being a solution of (1), the expression within the former bracket reduces to zero. Similarly, $y = \phi(x)$ being a solution of (2), the second group of terms is equal to X.

Hence (3) is a solution of the equation (2).

Theorem 2. *If* $y = y_1$, $y = y_2, \ldots\ldots, y = y_n$ *be integrals of the equation*

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0, \quad \ldots \quad (1)$$

then $y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$,

where $C_1, C_2, \ldots\ldots, C_n$ *are arbitrary constants, is also an integral of the equation* (1).

Substituting $y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$ in the left hand side of the equation (1), we get

$$C_1\left(\frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots\cdots + P_n y_1\right)$$
$$+ C_2\left(\frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots\cdots + P_n y_2\right)$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ C_n\left(\frac{d^n y_n}{dx^n} + P_1 \frac{d^{n-1} y_n}{dx^{n-1}} + \cdots\cdots + P_n y_n\right).$$

Now, since $y = y_1$, $y = y_2, \ldots\ldots, y = y_n$ are solutions of the given equation, each group of terms within the brackets is zero. This shows that

$$y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$$

is a solution of the equation (1).

Since this solution contains n arbitrary constants, it is the general solution of the equation (1).

From the above two theorems, we see that the general or complete solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X$$

consists of two parts. The first part is the general solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = 0$$

say, $$y = f_1(C_1, C_2, \ldots\ldots, C_n, x)$$

containing n arbitrary constants, and the second part

$$y = f_2(x)$$

say, is a solution of the equation under consideration and does not contain an arbitrary constant.

The first part, that is, the expression $f_1(C_1, C_2, \ldots\ldots, C_n, x)$ is called the *complementary function* (C. F.) and the second part, that is, $f_2(x)$ is called the *particular integral* (P. I).

The *complete* or *general solution* of the equation is thus

$$y = f_1(C_1, C_2, \ldots\ldots, C_n, x) + f_2(x) = C.F. + P.I.$$

In continuation of above theorems, we state below a few key facts about linear differential equations of second order. These can easily be verified as before.

(A) If a function $y_1(x)$ be a solution of a linear differential eqation, then the function $Cy_1(x)$, where C is an arbitrary constant, is also a solution of that equation.

(B) If the functions $y_1(x)$ and $y_2(x)$ be solutions of a linear differential equation, then the sum function $\{C_1 y_1(x) + C_2 y_2(x)\}$ is also a solution of that equation, where C_1 and C_2 are arbitrary constants.

If neither y_1 nor y_2 be a constant multiple of the other, then y_1 and y_2 form a *fundamental set of solutions* of the equation and $(C_1 y_1 + C_2 y_2)$ is called a *linear combination* of y_1 and y_2.

(C) If a linear differential equation with real coefficients has a complex solution $y(x) = u(x) + iv(x)$, then each of the real part $u(x)$ of this solution and the imaginary part $v(x)$ is also a solution of that equation.

(D) *Superposition Principle* : If y_p be a particular solution of $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f_1$ and $\overline{y}_p$ be that of $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f_2$, then $(y_p + \overline{y}_p)$ is a particular solution of $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f_1 + f_2$.

Note 1. Some authors use the symbols y_c and y_p to denote the complementary solution and the particular solution of the equation, so that the general solution is written as $y = y_c + y_p$.

Note 2. $y = C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n$ will be the general solution of the equation $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots + P_n y = 0$,

provided $y_1, y_2, \dots\dots, y_n$ are linearly independent*, that is, there does not exist a set of constants $a_1, a_2, \dots\dots, a_n$, at least one of which is non-zero, such that

$$a_1 y_1 + a_2 y_2 + \dots\dots + a_n y_n \equiv 0.$$

If these functions be linearly dependent (say $a_1 \neq 0$), then y_1 can be written in terms of the others as

$$y_1 = \frac{-(a_2 y_2 + a_3 y_3 + \dots\dots + a_n y_n)}{a_1}.$$

Hence it is clear that this solution can be put as

$$y = \left(C_2 - \frac{a_2 C_1}{a_1} \right) y_2 + \dots\dots + \left(C_n - \frac{a_n C_1}{a_1} \right) y_n,$$

which contains $(n-1)$ constants and hence is not the general solution.

A linear dependence of a pair of functions means that one of the functions can be obtained from the other by multiplying with a constant.

5.2. Differential operator D.

We use the symbol D for the differential operator $\frac{d}{dx}$, so that for $\frac{d^r y}{dx^r}$, we write $D^r y$.

If m_1 be a constant, then $(D - m_1) y \equiv \frac{dy}{dx} - m_1 y$.

The notation $(D - m_1)(D - m_2) y$ is defined to mean that y is operated first with $(D - m_2)$ and then the result is operated with $(D - m_1)$.

Thus, if m_1, m_2 be constants, then

$$(D - m_1)(D - m_2) y = (D - m_1) \left(\frac{dy}{dx} - m_2 y \right)$$

$$= \frac{d^2 y}{dx^2} - m_1 \frac{dy}{dx} - m_2 \frac{dy}{dx} + m_1 m_2 y$$

$$= \frac{d^2 y}{dx^2} - (m_1 + m_2) \frac{dy}{dx} + m_1 m_2 y.$$

* For details, see a later Chapter.

It can be easily verified that

$$(D - m_2)(D - m_1)y = \{D^2 - (m_1 + m_2)D + m_1 m_2\}y$$
$$= \frac{d^2y}{dx^2} - (m_1 + m_2)\frac{dy}{dx} + m_1 m_2 y.$$

Thus we see that if m_1 and m_2 be constants, then

$$(D - m_1)(D - m_2)y = (D - m_2)(D - m_1)y,$$

that is, operation is independent of the order in which the factors are used.

5.3. Solution of linear equations with constant coefficients.

Using the symbol $D\left(\equiv \frac{d}{dx}\right)$, the equation

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_n y = 0,$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants, can be written as

$$(D^n + P_1 D^{n-1} + \cdots\cdots + P_n)y = 0,$$

that is, $\quad f(D)y = 0, \qquad \ldots (1)$

where $\quad f(D) \equiv D^n + P_1 D^{n-1} + \cdots\cdots + P_n.$

Let $P_1, P_2, \ldots\ldots, P_n$ be real, so that the roots of the equation $f(m) = 0$ are either real or conjugate complex.

Here the degree of the equation $f(m) = 0$ is n. Let us assume that the polynomial equation $f(m) = 0$ has n real and distinct roots $m_1, m_2, \ldots\ldots, m_n$, so that (1) may be written as

$$(D - m_1)(D - m_2)\cdots\cdots(D - m_n)y = 0. \qquad \ldots (2)$$

The solution of any one of the equations

$$(D - m_1)y = 0, \ (D - m_2)y = 0, \ \ldots\ldots, (D - m_n)y = 0 \qquad \ldots (3)$$

is also a solution of the equation (2). For, if $\phi_2(x)$ be a solution of $(D - m_2)y = 0$, then putting $\phi_2(x)$ for y on the left hand expression of (2), we get

$$f(D)\phi_2 = (D - m_1)(D - m_3)\cdots\cdots(D - m_n)(D - m_2)\phi_2$$
$$= (D - m_1)(D - m_3)\cdots\cdots(D - m_n)(0)$$

$= 0$, since the operational factors are independent of the order in which they are used.

Thus $\phi_2(x)$ is a solution of the equation (1) and similar consideration can be made for the other equations in (3).

Now, if we integrate $(D-m)y=0$,

that is, $$\frac{dy}{dx}-my=0,$$

we get $y=Ce^{mx}$, where C is an arbitrary constant.

Hence the solutions of equations (3) are

$$y=C_1e^{m_1x},\ y=C_2e^{m_2x},\ \cdots\cdots,\ y=C_ne^{m_nx}, \quad \ldots \quad (4)$$

where $C_1, C_2, \ldots\ldots, C_n$ are arbitrary constants.

Each of these solutions being a solution of the equation (1), the general solution of the equation (1) is

$$y=C_1e^{m_1x}+C_2e^{m_2x}+\cdots\cdots+C_ne^{m_nx}. \quad \ldots \quad (5)$$

Since the constants $m_1, m_2, \ldots\ldots, m_n$ are distinct, the solutions (4) are linearly independent and hence (5) is the general solution of (1).

It should be noted that the n distinct numbers $m_1, m_2, \ldots\ldots, m_n$ can be found by solving, for m, the equation

$$m^n+P_1m^{n-1}+\cdots\cdots+P_n=0,$$

which is obtained by substituting e^{mx} for y in (1), since $e^{mx}\neq 0$.

This equation is called the *auxiliary equation* or *characteristic equation*.

In this case, the roots of the auxiliary equation are real and distinct.

5.4. Case of the auxiliary equation having equal roots.

When two roots of the auxiliary equation are equal, that is, $m_1=m_2=m$ (say), then the solution obtained in the previous article becomes

$$y=(C_1+C_2)e^{mx}+C_3e^{m_3x}+\cdots\cdots+C_ne^{m_nx}$$
$$=Ce^{mx}+C_3e^{m_3x}+\cdots\cdots+C_ne^{m_nx}, \text{ where } C_1+C_2=C.$$

This is no longer a general solution, since the number of arbitrary constants is now $(n-1)$ and not n.

The corresponding part of the solution is, in fact, the solution of

$$(D-m)^2y=0$$
or, $$(D-m)(D-m)y=0$$
or, $$(D-m)u=0,$$

where u is put for $(D-m)y$.

Solution of this equation is $u = C_2 e^{mx}$.

Putting this value of u, we get

$$(D - m) y = u = C_2 e^{mx}, \text{ that is, } \frac{dy}{dx} - my = C_2 e^{mx},$$

which is a linear equation of first order whose integrating factor is e^{-mx} and the solution is

$$y e^{-mx} = \int C_2 e^{mx} . e^{-mx} dx = C_1 + C_2 x .$$

Therefore $$y = (C_1 + C_2 x) e^{mx},$$

in which there are two constants C_1 and C_2.

Thus the general solution in this case is

$$y = (C_1 + C_2 x) e^{mx} + C_3 e^{m_3 x} + \cdots\cdots + C_n e^{m_n x}.$$

Cor. If the auxiliary equation has r equal roots m, then the general solution is

$$y = \left(C_1 + C_2 x + \cdots + C_r x^{r-1}\right) e^{mx} + C_{r+1} e^{m_{r+1} x} + \cdots + C_n e^{m_n x}.$$

5.5. Case of the auxiliary equation having complex roots.

If the auxiliary equation has a pair of complex roots, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the corresponding part of the solution is

$$C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$
$$= e^{\alpha x} (C_1 e^{i\beta x} + C_2 e^{-i\beta x})$$
$$= e^{\alpha x} \{ C_1 (\cos \beta x + i \sin \beta x) + C_2 (\cos \beta x - i \sin \beta x) \}$$
$$= e^{\alpha x} \{ (C_1 + C_2) \cos \beta x + i (C_1 - C_2) \sin \beta x \}$$
$$= e^{\alpha x} (A \cos \beta x + B \sin \beta x),$$

in which $A = C_1 + C_2$ and $B = i(C_1 - C_2)$ are arbitrary constants.

If the above pair of complex roots occurs twice in the auxiliary equation, then the corresponding part of the solution is

$$(C_1 + C_2 x) e^{(\alpha + i\beta)x} + (C_3 + C_4 x) e^{(\alpha - i\beta)x},$$

which reduces to

$$e^{\alpha x} \{ (A_1 + A_2 x) \cos \beta x + (B_1 + B_2 x) \sin \beta x \},$$

A_1, A_2, B_1, B_2 being arbitrary constants.

Note. The methods of solution given above are due to Euler and D'Alembert.

5.6. Illustrative Examples.

Ex. 1. *Solve the equation* $2\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$.

Let $y = e^{mx}$ be a solution of the above equation.

Then the equation becomes

$$(2m^3 - 7m^2 + 7m - 2)e^{mx} = 0.$$

Since $e^{mx} \neq 0$, we get

$$2m^3 - 7m^2 + 7m - 2 = 0$$

or, $$(m-1)(m-2)(2m-1) = 0.$$

Therefore $m = 1, 2, \frac{1}{2}$.

Hence the general solution is

$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{\frac{1}{2}x}$, where C_1, C_2, C_3 are arbitrary constants.

Ex. 2. *Find the general solution of the equation*

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$$

Let $y = e^{mx}$ be a solution of the above differential equation ;

then we have $(m^4 - m^3 - 9m^2 - 11m - 4)e^{mx} = 0$.

Therefore $m^4 - m^3 - 9m^2 - 11m - 4 = 0$, since $e^{mx} \neq 0$

or, $$(m+1)^3(m-4) = 0,$$

giving $m = -1, -1, -1, 4$.

Therefore the general solution is

$$y = (C_1 + C_2 x + C_3 x^2)e^{-x} + C_4 e^{4x},$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Ex. 3. *Solve the equation* $\frac{d^4y}{dx^4} + a^4 y = 0$.

Introducing the differential operator $D\left(\equiv \frac{d}{dx}\right)$, the given equation can be written as

$$(D^4 + a^4)y = 0,$$

so that the auxiliary equation is $m^4 + a^4 = 0$,

giving $m = -\frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$ and $m = \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$.

Hence the general solution is

$$y = \left(C_1 \cos \frac{a}{\sqrt{2}} x + C_2 \sin \frac{a}{\sqrt{2}} x \right) e^{-\frac{a}{\sqrt{2}} x} + \left(C_3 \cos \frac{a}{\sqrt{2}} x + C_4 \sin \frac{a}{\sqrt{2}} x \right) e^{\frac{a}{\sqrt{2}} x},$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Ex. 4. *Solve the equation* $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$, *where*

$$D \equiv \frac{d}{dx}.$$

Here the auxiliary equation is $(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$.

Therefore $m = \pm i, \pm i, \pm i$ and $m = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$.

Hence the general solution is

$$y = (C_1 + C_2 x + C_3 x^2) \cos x + (C_4 + C_5 x + C_6 x^2) \sin x$$

$$+ \left\{ (C_7 + C_8 x) \cos \frac{\sqrt{3}}{2} x + (C_9 + C_{10} x) \sin \frac{\sqrt{3}}{2} x \right\} e^{-\frac{1}{2} x},$$

in which $C_1, C_2, \ldots\ldots, C_{10}$ are arbitrary constants.

Note that the general solution contains ten arbitrary constants, which is the same as the order of the given equation.

Note. The values of the arbitrary constants and hence the particular solution of the equation can be determined from given conditions.

5.7. Symbolic operator $\dfrac{1}{f(D)}$.

We use the expression $\dfrac{1}{f(D)}X$ to denote a function of x, which does not contain any arbitrary constant and which gives X when operated with $f(D)$. Thus, since

$$(D^2 - D)(x^2 - x) = 3 - 2x,$$

we have
$$\frac{1}{D^2 - D}(3 - 2x) = x^2 - x.$$

The operator $\frac{1}{f(D)}$, according to this definition, is the inverse of the operator $f(D)$. If $f(D) = D$, then we have

$$\frac{1}{f(D)} X = \frac{1}{D} X = \int X \, dx .$$

For our future use, we attempt to find v, which is obtained by inversely operating on X with the factor $(D - a)$, that is,

$$v = \frac{1}{D - a} X ,$$

in which X is a function of x only and a is a constant. This is, according to the definition,

$$(D - a) v = X$$

or,
$$\frac{dv}{dx} - av = X ,$$

which is a linear equation of first order in v and whose integrating factor is e^{-ax}. Therefore its solution is given by

$$v = Ae^{ax} + e^{ax} \int X e^{-ax} \, dx .$$

Now, as v, by definition, will remain free from any arbitrary constant, we have

$$v = e^{ax} \int X e^{-ax} \, dx .$$

This result will be found useful in the discussion of the general method of finding the particular integral of an equation.

Note. If, in particular, $X = e^{ax}$, then $v = xe^{ax}$.

5.8. General method of finding the particular integral.

Consider the linear equation with constant coefficients

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X ,$$

where X is a function of x only. In symbolic notation, this can be written as

$$f(D) \, y = X ,$$

where
$$f(D) \equiv D^n + P_1 D^{n-1} + \cdots\cdots + P_n$$

and the particular integral is thus $\frac{1}{f(D)} X$.

We can now evaluate $\frac{1}{f(D)} X$ by any one of the following methods:

(i) Let $f(D)$ can be broken up into linear factors, say,

$$f(D) = (D - m_1)(D - m_2) \ldots\ldots\ldots\ldots (D - m_n),$$

the factors being placed in any order.

Then the particular integral is

$$\frac{1}{D - m_1} \cdot \frac{1}{D - m_2} \ldots\ldots\ldots\ldots \frac{1}{D - m_n} X.$$

This expression is defined to mean that X is first inversely operated upon with $(D - m_n)$, then the result is inversely operated upon with $(D - m_{n-1})$ and so on until all the factors are similarly utilised.

After the first operation, it becomes

$$\frac{1}{D - m_1} \frac{1}{D - m_2} \ldots\ldots\ldots\ldots e^{m_n x} \int X e^{-m_n x} dx.$$

Then operating with the second and remaining factors in succession, we get the particular integral as

$$e^{m_1 x} \int e^{(m_2 - m_1)x} \int \ldots\ldots\ldots \int X e^{-m_n x} (dx)^n.$$

(ii) Let $\frac{1}{f(D)}$ can be resolved into partial fractions, say,

$$\frac{1}{f(D)} = \frac{N_1}{D - m_1} + \frac{N_2}{D - m_2} + \ldots\ldots\ldots + \frac{N_n}{D - m_n},$$

$N_1, N_2, \ldots\ldots\ldots, N_n$ being constants.

Therefore

$$\frac{1}{f(D)} X = \frac{N_1}{D - m_1} X + \frac{N_2}{D - m_2} X + \ldots\ldots\ldots + \frac{N_n}{D - m_n} X$$

$$= N_1 e^{m_1 x} \int X e^{-m_1 x} dx + \ldots\ldots\ldots + N_n e^{m_n x} \int X e^{-m_n x} dx.$$

5.9. Particular integral by short methods.

The general method of finding the particular integral is a laborious calculation. There are short methods, for finding them for some functions, which we shall explain now.

(*i*) *Particular integral for* $X = e^{ax}$, *a being a constant.*

The equation here is $f(D)\,y = e^{ax}$,

so that the particular integral is $\dfrac{1}{f(D)}\,e^{ax}$.

We have

$De^{ax} = ae^{ax},\ D^2e^{ax} = a^2e^{ax}, \ldots, D^{n-1}e^{ax} = a^{n-1}e^{ax},\ D^ne^{ax} = a^ne^{ax}.$

Therefore

$$\begin{aligned} f(D)\,e^{ax} &= (D^n + P_1D^{n-1} + \cdots\cdots + P_{n-1}D + P_n)\,e^{ax} \\ &= (a^n + P_1a^{n-1} + \cdots\cdots + P_{n-1}a + P_n)\,e^{ax} \\ &= f(a)\,e^{ax}. \end{aligned}$$

Operating both sides with $\dfrac{1}{f(D)}$, we get

$$\frac{1}{f(D)}\{f(D)\,e^{ax}\} = \frac{1}{f(D)}\{f(a)\,e^{ax}\}.$$

Now, since $f(D)$ and $\dfrac{1}{f(D)}$ are operators inverse to one another and $f(a)$ is only an algebraic multiplier, it reduces to

$$e^{ax} = f(a)\cdot\frac{1}{f(D)}\,e^{ax},$$

whence we have $\qquad \dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{f(a)}\,e^{ax}$, provided $f(a) \neq 0$.

If $f(D)$ contains a factor $(D - a)$, then this method fails and we proceed in the following way :

Since $(D - a)$ is a factor of $f(D)$, let $f(D) = (D - a)\,\phi(D)$.

Then $\dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{D-a}\,\dfrac{1}{\phi(D)}\,e^{ax} = \dfrac{1}{D-a}\,\dfrac{1}{\phi(a)}\,e^{ax}$, provided $\phi(a) \neq 0$

$$= \frac{xe^{ax}}{\phi(a)} \qquad [\text{ cf. Note of Art. 5.7 }]$$

If $(D - a)^2$ be a factor of $f(D)$, let $f(D) = (D - a)^2\,\psi(D)$.

Then $\dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{(D-a)^2}\,\dfrac{1}{\psi(D)}\,e^{ax} = \dfrac{1}{(D-a)^2}\,\dfrac{1}{\psi(a)}\,e^{ax}$,

provided $\psi(a) \neq 0$

$$= \frac{x^2e^{ax}}{2\,\psi(a)}.$$

Same procedure will be followed when $(D - a)^r$ is a factor of $f(D)$, r being a positive integer.

(ii) *Particular integral for $X = x^m$, m being a positive integer.*

In order to evaluate $\frac{1}{f(D)} x^m$,

expand $\{f(D)\}^{-1}$ and arrange the terms in ascending powers of D and operate on x^m. The result will be the particular integral corresponding to x^m.

It should be noticed that terms of the expansion beyond the m-th power of D need not be written, since $D^{m+1} x^m = 0$.

(iii) *Particular integral for $X = \sin ax$ or $\cos ax$.*

Let us evaluate $\frac{1}{f(D)} \sin ax$.

We have
$$D \sin ax = a \cos ax,$$
$$D^2 \sin ax = -a^2 \sin ax,$$
$$D^3 \sin ax = -a^3 \cos ax,$$
$$D^4 \sin ax = a^4 \sin ax.$$

In general, $(D^2)^n \sin ax = (-a^2)^n \sin ax$.

Now, if $f(D)$ contains only even powers of D and we denote it by $\phi(D^2)$, then it is obvious that

$$\phi(D^2) \sin ax = \phi(-a^2) \sin ax.$$

Operating on both sides with $\frac{1}{\phi(D^2)}$, we get

$$\sin ax = \frac{1}{\phi(D^2)} \{\phi(-a^2) \sin ax\}.$$

Since $\phi(-a^2)$ is an algebraic multiplier, we get

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax, \text{ provided } \phi(-a^2) \neq 0.$$

Similarly, we get

$$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax, \quad \text{provided } \phi(-a^2) \neq 0.$$

More generally, we have

$$\frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b),\ \phi(-a^2) \neq 0$$

and $$\frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b),\ \phi(-a^2) \neq 0.$$

The above results do not hold in case $\phi(-a^2) = 0$. This will happen, if $f(D)$ contains a factor $(D^2 + a^2)$. In such cases, the general method is applied to find the particular integral.

We present a special method here for such cases.

In such cases, instead of computing the particular integral for $\sin ax$ or $\cos ax$, we calculate the particular integral for $(\cos ax + i \sin ax)$, that is, for e^{iax}. Thus

$$\frac{1}{D^2 + a^2} (\cos ax + i \sin ax) = \frac{1}{D^2 + a^2} e^{iax}$$

$$= \frac{1}{(D + ia)(D - ia)} e^{iax}$$

$$= \frac{e^{iax}}{2ai} \frac{1}{D + ia - ia} 1$$

$$= \frac{e^{iax}}{2ai} \frac{1}{D} 1$$

$$= \frac{x e^{iax}}{2ai} = \frac{x}{2ai} (\cos ax + i \sin ax)$$

$$= \frac{x \sin ax}{2a} - i \frac{x \cos ax}{2a}.$$

Equating the real and the imaginary parts from both sides, we get

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$$

and $$\frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax.$$

(iv) Particular integral for $X = e^{ax} V$, V being any function of x.

We are to evaluate $\frac{1}{f(D)} (e^{ax} V)$.

Let V_1 be a function of x defined by $V_1 = \frac{1}{f(D + a)} V$. ... (1)

We have $D(e^{ax}V_1) = e^{ax}DV_1 + ae^{ax}V_1 = e^{ax}(D+a)V_1$,

$$D^2(e^{ax}V_1) = ae^{ax}(D+a)V_1 + e^{ax}D(D+a)V_1$$
$$= e^{ax}(D+a)^2V_1.$$

In general, by successive differentiation, we get

$$D^n(e^{ax}V_1) = e^{ax}(D+a)^nV_1.$$

Therefore $f(D)(e^{ax}V_1) = e^{ax}f(D+a)V_1$. ... (2)

Putting (1) in (2), we get

$$f(D)\left\{e^{ax}\frac{1}{f(D+a)}V\right\} = e^{ax}V.$$

Now operating both sides of this equation with $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.$$

(v) *Particular integral for $X = xV$, V being any function of x.*

We are to evaluate $\frac{1}{f(D)}(xV)$.

Let V_1 be a function of x defined by $V_1 = \frac{1}{f(D)}V$. ... (1)

We have $D(xV_1) = xDV_1 + V_1$,

$$D^2(xV_1) = xD^2V_1 + 2DV_1 = xD^2V_1 + \left(\frac{d}{dD}D^2\right)V_1,$$

....

$$D^n(xV_1) = xD^nV_1 + nD^{n-1}V_1, \text{ by Leibnitz's theorem}$$
$$= xD^nV_1 + \left(\frac{d}{dD}D^n\right)V_1.$$

Hence $f(D)(xV_1) = xf(D)V_1 + f'(D)V_1$, ... (2)

where $f'(D) \equiv \frac{d}{dD}\{f(D)\}$.

Putting (1) in (2), we get

$$f(D)\left\{x\frac{1}{f(D)}V\right\} = xV + f'(D)\frac{1}{f(D)}V.$$

Operating all the terms of this equation with $\frac{1}{f(D)}$, we obtain

$$x\frac{1}{f(D)}V = \frac{1}{f(D)}(xV) + \frac{1}{f(D)}\left\{f'(D)\frac{1}{f(D)}V\right\}$$
$$= \frac{1}{f(D)}(xV) + f'(D)\frac{1}{\{f(D)\}^2}V.$$

Transposing, we get

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - f'(D)\frac{1}{\{f(D)\}^2}V$$

$$= x\frac{1}{f(D)}V + \left[\frac{d}{dD}\left\{\frac{1}{f(D)}\right\}\right]V.$$

The particular integral corresponding to $X = x^m V$, where m is a positive integer can be obtained by repeated application of this method.

5.10. Illustrative Examples.

Ex. 1. *Solve* : $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$.

Introducing the differential operator $D\left(\equiv \frac{d}{dx}\right)$, the given equation can be written as

$$(D^3 + 3D^2 + 2D)y = x^2$$

The auxiliary equation is $m^3 + 3m^2 + 2m = 0$

$$\text{or,} \quad m(m+1)(m+2) = 0.$$

Therefore $\quad m = 0, -1, -2.$

Hence the complementary function (y_c) is

$$A + Be^{-x} + Ce^{-2x},$$

where A, B, C are arbitrary constants.

The particular integral (y_p) is

$$\frac{1}{D^3 + 3D^2 + 2D}x^2 = \frac{1}{2D}\left(1 + \frac{3D + D^2}{2}\right)^{-1}x^2$$

$$= \frac{1}{2D}\left\{1 - \frac{3D + D^2}{2} + \left(\frac{3D + D^2}{2}\right)^2 - \cdots\cdots\right\}x^2$$

$$= \frac{1}{2D}\left\{1 - \frac{3D}{2} - \frac{D^2}{2} + \frac{9D^2 + 6D^3 + D^4}{4} - \cdots\cdots\right\}x^2$$

$$= \frac{1}{2D}\left(1 - \frac{3D}{2} + \frac{7}{4}D^2 - \cdots\cdots\cdots\right)x^2$$

$$= \frac{1}{2}\left(\frac{1}{D} - \frac{3}{2} + \frac{7}{4}D\right)x^2 = \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{7}{4}\cdot 2x\right)$$

$$= \frac{1}{12}(2x^3 - 9x^2 + 21x).$$

Hence the complete solution is

$$y = y_c + y_p = A + Be^{-x} + Ce^{-2x} + \frac{1}{12}(2x^3 - 9x^2 + 21x).$$

Ex. 2. *Solve :* $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x$.

The given equation, in terms of the operator $D\left(\equiv \dfrac{d}{dx}\right)$, is

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x.$$

The auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0, \text{ so that } m = 1, 1 \pm i.$$

The complementary function is

$$y_c = Ae^x + (B\cos x + C\sin x)e^x,$$

where A, B, C are arbitrary constants.

The particular integral is

$$y_p = \frac{1}{(D^3 - 3D^2 + 4D - 2)}(e^x + \cos x)$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)}e^x + \frac{1}{(D-1)(D^2 - 2D + 2)}\cos x$$

$$= \frac{1}{(D-1)(1 - 2 + 2)}e^x + \frac{1}{(D-1)(-1 - 2D + 2)}\cos x$$

$$= e^x \frac{1}{(D + 1 - 1)}1 + \frac{1}{-2D^2 + 3D - 1}\cos x$$

$$= e^x \cdot x + \frac{1}{-2(-1) + 3D - 1}\cos x$$

$$= xe^x + \frac{1}{3D + 1}\cos x = xe^x + \frac{3D - 1}{9D^2 - 1}\cos x$$

$$= xe^x + \frac{3D - 1}{9(-1) - 1}\cos x = xe^x - \frac{1}{10}(3D - 1)\cos x$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x).$$

Hence the complete solution is

$$y = y_c + y_p$$

$$= Ae^x + (B\cos x + C\sin x)e^x + xe^x + \frac{1}{10}(3\sin x + \cos x).$$

5.11. Formation of linear differential equations, whose solutions are specified.

While solving linear differential equations with constant coefficients, we observed that if the auxiliary equation $f(m) = 0$ had a root $m = \alpha$, then the operator $f(D)$ had a factor $(D - \alpha)$ and a term such as $Ae^{\alpha x}$ occurred in the general solution of the equation. In the same way, we observed that $Bxe^{\alpha x}$ appeared in the solution only when $f(D)$ contained a factor $(D-\alpha)^2$, $Cx^2e^{\alpha x}$ only when $f(D)$ contained $(D-\alpha)^3$ and so on. $A, B, C, \ldots\ldots$ are independent of x. We also observed that terms as $Ae^{\alpha x}\sin\beta x$ or $Be^{\alpha x}\cos\beta x$, which occured in the solution, correspond to roots $m=\alpha\pm i\beta$ of the auxiliary equation or to factor $\{(D-\alpha)^2+\beta^2\}$ in the operator $f(D)$.

With these facts in our possession, we try to form homogeneous linear differential equations, whose solutions are specified, assuming the superposition principle. For that, let the given function be $(Ae^{2x}+Bx)$, with no restrictions on A and B.

For the term Ae^{2x}, we know that there will be a root $m=2$ of the auxiliary equation and a factor $(D-2)$ of the operator $f(D)$. The term Bx will appear, if the auxiliary equation has a double root $m=0, 0$ and a factor D^2 in $f(D)$. Hence we see that the homogeneous linear equation

$$D^2(D-2)\,y=0$$

that is, $$(D^3-2D^2)\,y=0 \qquad \ldots (1)$$

has $y=Ae^{2x}+Bx+C$ as its general solution and hence $y=Ae^{2x}+Bx$ will be a particular solution of (1), of which A and B are constants.

Again, if the specified function be

$$y=10+5xe^{x}+\sin x, \qquad \ldots (2)$$

then we see that the term 10 is associated with the root $m=0$ of the auxiliary equation, the term $5xe^{x}$ is associated with the double root $m=1, 1$ of the auxiliary equation and the term $\sin x$ is associated with the pair of imaginary roots $m=0\pm i$. Hence the auxiliary equation is

$$m(m-1)^2(m^2+1)=0$$

or, $$m^5-2m^4+2m^3-2m^2+m=0.$$

The differential equation corresponding to this auxiliary equation is

$$(D^5-2D^4+2D^3-2D^2+D)\,y=0, \qquad \ldots (3)$$

the general solution of which is

$$y=A+(B+Cx)\,e^{x}+E\sin x+F\cos x.$$

With appropriate choice of the constants

$$A = 10, \ B = 0, \ C = 5, \ E = 1, \ F = 0,$$

the given relation (2) becomes a particular solution of the differential equation (3).

If again the specified function be

$$y = 7xe^{x} \cos 3x \qquad \ldots \quad (4)$$

and if we are to find the homogeneous linear differential equation with real constant coefficients of which this function is a particular solution, then we observe that the desired equation will have its auxiliary equation $f(m) = 0$, whose roots will be $m = 1 \pm 3i$ and $1 \pm 3i$. Hence the auxiliary equation will be

$$\{(m-1)^2 + 9\}^2 = 0$$

or,

$$m^4 - 4m^3 + 24m^2 - 40m + 100 = 0 .$$

Thus the desired equation will be

$$(D^4 - 4D^3 + 24D^2 - 40D + 100)\, y = 0, \qquad \ldots \quad (5)$$

the general solution of which will be

$$y = e^{x}(A + Bx) \cos 3x + e^{x}(E + Fx) \sin 3x .$$

With appropriate choice of the constants

$$A = E = F = 0 \text{ and } B = 7,$$

the given relation (4) becomes a particular solution of the differential equation (5).

5.12 Method of undetermined coefficients.

Let us consider the linear differential equation of n-th order

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X, \qquad \ldots \quad (1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants and X is a function of x only.

In symbolic notation, this can be written as

$$f(D)\, y = X,$$

where

$$f(D) \equiv D^n + P_1 D^{n-1} + \cdots\cdots + P_n .$$

We know that the general solution of the equation (1) may be expressed as

$$y = y_c + y_p,$$

where y_c is the complementary function, that is, the general solution of the corresponding homogeneous equation $f(D)\,y = 0$ and y_p is the particular integral, that is, any solution of the equation (1) containing no arbitrary constants.

The method of undetermined coefficients is one of the procedures for determining y_p, when X is an exponential, a polynomial, a sine or cosine, or some combination of such functions only.

(*i*) *Particular integral for $X = e^{ax}$, a being a constant.*

The equation here is $\quad f(D)\,y = e^{ax}$. $\quad\ldots\quad$ (2)

Since the derivatives of e^{ax} are constant multiples of e^{ax}, it is reasonable to guess that the particular integral might also be a constant multiple of e^{ax} and we assume that

$$y_p = Ae^{ax} \qquad \ldots \quad (3)$$

might be a particular solution of the equation (2).

Here A is a constant (called the undetermined coefficient) to be determined, such that (3) actually satisfies (2).

Thus we get

$$A\,f(a)\,e^{ax} = e^{ax}, \text{ giving } A = \frac{1}{f(a)}, \text{ provided } f(a) \neq 0.$$

If $f(a) = 0$, that is, if a be a root of the auxiliary equation $f(m) = 0$, then (3) reduces the left hand side of (2) to zero. But the right hand side of (2) being different from zero, (3) possibly cannot satisfy (2), as it stands. In this case, we take

$$y_p = A\,x\,e^{ax}$$

as a trial solution.

If a be a double root of the auxiliary equation, then we take $y_p = Ax^2 e^{ax}$ as a trial solution. In general, if a be a multiple root of the auxiliary equation of multiplicity $r\,(< n)$, then we take $y_p = ax^r e^{ax}$ as a trial solution.

(ii) *Particular integral for* $X = a_0 x^k + a_1 x^{k-1} + \cdots\cdots + a_k$, *where k is a positive integer and* $a_0, a_1, \ldots\ldots, a_k$ *are constants.*

Here the equation is

$$f(D)\, y = a_0 x^k + a_1 x^{k-1} + \cdots\cdots + a_k. \qquad \ldots \quad (4)$$

Since the derivative of a polynomial is also a polynomial, we assume that

$$y_p = A_0 x^k + A_1 x^{k-1} + \cdots\cdots + A_k \qquad \ldots \quad (5)$$

might be a particular solution of the equation (4), if $P_n \neq 0$.

Here $A_0, A_1, \ldots\ldots, A_k$ are constants (undetermined coefficients) to be determined, such that (5) satisfies (4). Substituting (5) in (4) and equating like powers of x from both sides, we get the values of

$$A_0, A_1, \ldots\ldots, A_k.$$

If $P_n = 0$ (that is, if there be no term containing y), then putting (5) in (4), we see that the highest power of x on the left hand side of (4) is x^{k-1} but that on the right hand side of (4) is x^k. So, in this case, we take $y_p = x(A_0 x^k + A_1 x^{k-1} + \cdots\cdots + A_k)$.

If $P_n = 0$ and $P_{n-1} = 0$, then we take

$$y_p = x^2 (A_0 x^k + A_1 x^{k-1} + \cdots\cdots + A_k).$$

In general, if the last r P's be zero but $P_{n-r} \neq 0$, then we take

$$y_p = x^r (A_0 x^k + A_1 x^{k-1} + \cdots\cdots + A_k).$$

(iii) *Particular integral for* $X = \sin ax$ *or* $\cos ax$.

Let the equation be $f(D)\, y = \sin ax$. $\qquad \ldots \quad (6)$

Since the derivatives of $\sin ax$ are constant multiples of $\sin ax$ and $\cos ax$, we take

$$y_p = A \sin ax + B \cos ax \qquad \ldots \quad (7)$$

as a trial solution of the equation (6), provided (7) does not satisfy the homogeneous equation $f(D)\, y = 0$. Here A and B are constants (undetermined coefficients) and are obtained by putting (7) in (6) and equating the resulting coefficients of $\sin ax$ and $\cos ax$ from both sides.

If (7) satisfies the equation $f(D)\,y = 0$, (that is, if (7) be a p[art] of y_c), then we take

$$y_p = x\,(A \sin ax + B \cos ax)$$

as a trial solution.

Note. If X be a linear combination of e^{ax}, $\sin bx$ and $\Sigma\, a_n x^{k-n}$, then particular integral is obtained by following the superposition principle.

If X be of the form $e^{ax} \sin bx$ or $e^{ax}(\Sigma\, a_n x^{k-n})$ or $(\Sigma\, a_n x^{k-n}) \sin$ or $e^{ax}(\Sigma\, a_n x^{k-n}) \sin bx$, then y_p is modified acccordingly. For examp[le] if $X = e^{ax} \sin bx$, where a is not a root of the auxiliary equation and $\sin bx$ not a part of y_c, then we take $y_p = (A_1 e^{ax})\,(B_1 \sin bx + C_1 \cos bx$ that is, $y_p = e^{ax}(A \sin bx + B \cos bx)$.

5.13. Illustrative Examples.

Ex. 1. *Obtain linear differential equation with real constant coefficient that is satisfied by the following functions :*

(*i*) $y = x^2 - 8 \sin 4x$; (*ii*) $y = 4e^{-x} \cos 3x + 12e^{-x} \sin 3x$.

(*i*) The term x^2 will appear, if the auxiliary equation has a multiple roo[t] of multiplicity 3 and $m = 0, 0, 0$. The term $\sin 4x$ will be given by a pair o[f] imaginary roots $0 \pm 4i$. The corresponding auxiliary equation is

$$m^3(m^2 + 16) = 0, \text{ that is, } m^5 + 16m^3 = 0.$$

Hence the given function is a solution of the homogeneous linea[r] differential equation

$$(D^5 + 16D^3)\,y = 0. \qquad \ldots (1)$$

Note. The general solution of the equation (1) is

$$y = A + Bx + Cx^2 + E \sin 4x + F \cos 4x.$$

The choice of the constants $A = B = F = 0$, $C = 1$ and $E = -8$ justifies that the given function is a particular solution of the equation (1).

(*ii*) The roots of the auxiliary equation corresponding to the terms of the given function are $(-1 \pm 3i)$.

The auxiliary equation is thus $\quad (m + 1)^2 + 9 = 0$

or, $\quad m^2 + 2m + 10 = 0$.

The corresponding linear homogeneous differential equation with constant coefficients is

$$(D^2 + 2D + 10)\,y = 0.$$

CHAPTER VI

HOMOGENEOUS LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

6.1. Homogeneous linear equations.

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X, \quad \ldots (1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants and X is either a constant or a function of x only is called a *homogeneous linear differential equation*. This is also known as *Euler-Cauchy type of equations.*

Equations of this type are solved by transforming them to equations with constant coefficients through a change of the independent variable x to z by the relation

$$x = e^z, \quad \text{that is}, \quad z = \log x.$$

When this change is effected, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right),$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^n y}{dx^n} = \frac{1}{x^n}\left\{\frac{d^n y}{dz^n} - \frac{n(n-1)}{2}\frac{d^{n-1}y}{dz^{n-1}} + \cdots\cdots + (-1)^{n-1}(n-1)!\frac{dy}{dz}\right\}.$$

We use the symbol D' for the differential operator $\frac{d}{dz}$.

Thus $D' \equiv \frac{d}{dz}$ and $D'^r \equiv \frac{d^r}{dz^r}$. Also $D' \equiv x\frac{d}{dx}$.

Putting this differential operator D' for $\frac{d}{dz}$, we get

$$x\frac{dy}{dx} = D'y,$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)y,$$

$$x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y,$$

....

....

$$x^n\frac{d^ny}{dx^n} = D'(D'-1)(D'-2)\ldots\ldots(D'-n+1)y.$$

Substituting these relations in (1), we get the transformed equation as

$$\{D'(D'-1)\cdots\cdots(D'-n+1)$$
$$+P_1D'(D'-1)\cdots\cdots(D'-n+2)+\cdots\cdots+P_n\}y = Z, \quad \ldots (2)$$

where Z is a function of z into which X is transformed by the substitution $x = e^z$.

This is an equation with constant coefficients and can be easily solved. If this equation (2) be written in the form

$$f(D')y = Z, \quad \ldots (3)$$

where $f(D') \equiv D'(D'-1)\cdots\cdots(D'-n+1)$

$$+P_1D'(D'-1)\cdots\cdots(D'-n+2)+\cdots\cdots+P_n,$$

then the complementary function will be given by different functions as determined by the roots of the auxiliary equation $f(m) = 0$, as in the previous Chapter.

The particular integral will be given by

$$\frac{1}{f(D')}Z$$

and can be evaluated by applying the methods discussed in the previous Chapter.

6.2. Alternative method to find complementary function..

Instead of making any transformation, we can find directly the complementary function of the equation

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X. \quad \ldots \quad (1)$$

If we put x^m for y, then the left hand side of the equation (1) becomes

$$\{m(m-1)(m-2) \cdots\cdots (m-n+1)$$
$$+ P_1 m(m-1) \cdots\cdots (m-n+2) + \cdots\cdots + P_n\} x^m.$$

Now, if

$$f(m) = m(m-1)(m-2) \cdots\cdots (m-n+1)$$
$$+ P_1 m(m-1) \cdots (m-n+2) + \cdots\cdot + P_n = 0, \quad \ldots \quad (2)$$

then the substitution of x^m for y makes the left hand side of (1) vanish. This assures that x^m is a part of the complementary function of the solution of (1).

The degree of the equation $f(m) = 0$ is n.

Now, if $m_1, m_2, \ldots\ldots, m_n$ be the n distinct roots of (2), then the complementary function is $C_1 x^{m_1} + C_2 x^{m_2} + \cdots\cdots + C_n x^{m_n}$, where $C_1, C_2, \ldots\ldots, C_n$ are arbitrary constants.

It is observed that the function $f(m)$ is the same function of m as is $f(D')$ of D' of the previous article. Therefore corresponding to an x^{m_1} of the C. F. of the solution of the equation (1), there is an $e^{m_1 z}$ of the C. F. of the solution of the equation (1) of the previous article. Hence, as has been seen, the C. F. of the solution of the equation (1) can be obtained by putting $\log x$ for z in the C. F. of the solution of the equation (2) of the previous article.

6.3. Alternative method to find particular integral.

We use the symbol D' (some authors use θ) for $\frac{d}{dz}$, that is, for $x\frac{d}{dx}$, that is, for xD.

If $f(D')$ can be expressed as

$$f(D') = (D' - \alpha_1)(D' - \alpha_2) \ldots\ldots (D' - \alpha_n),$$

then $\frac{1}{f(D')} X$ becomes

$$\frac{1}{D' - \alpha_1} \frac{1}{D' - \alpha_2} \ldots\ldots \frac{1}{D' - \alpha_n} X,$$

in which the operations effected by the factors will be taken in succession beginning from the extreme right factor. The final result will give the particular integral.

It is to be noted that the factors are commutative.

If the operator $\frac{1}{f(D')}$ can be broken up into partial fractions, then

$$\frac{1}{f(D')} X \text{ becomes } \left(\frac{N_1}{D' - \alpha_1} + \frac{N_2}{D' - \alpha_2} + \ldots\ldots + \frac{N_n}{D' - \alpha_n} \right) X,$$

$N_1, N_2, \ldots\ldots, N_n$ being constants.

The sum of the results effected by each fraction on X will give the particular integral.

Thus we see that in both the cases we are to operate X with the operation of the form $\frac{1}{D' - \alpha}$.

Let $$\frac{1}{D' - \alpha} X = y,$$

so that operating both sides with $(D' - \alpha)$, we get

$$(D' - \alpha) y = X$$

or, $$x \frac{dy}{dx} - \alpha y = X$$

or, $$\frac{dy}{dx} - \frac{\alpha}{x} y = X x^{-1},$$

which is a linear differential equation of the first order.

Integrating, we get

$$y e^{-\alpha \log x} = \int x^{-\alpha - 1} X \, dx$$

or, $$y = x^{\alpha} \int x^{-\alpha - 1} X \, dx.$$

Thus we have $\frac{1}{D'-\alpha}X = y = x^{\alpha}\int x^{-\alpha-1}X\,dx$.

Hence, when $f(D') = (D'-\alpha_1)(D'-\alpha_2)\ldots(D'-\alpha_n)$ and $\frac{1}{f(D')}$ can be expressed in factorial form, then the particular integral will be given by

$$x^{\alpha_1}\int x^{\alpha_2-\alpha_1-1}\int \ldots\ldots \int x^{-\alpha_n-1}X(dx)^n.$$

If, again, $\frac{1}{f(D')}$ can be expressed as a sum of partial fractions as above, then the particular integral is given by

$$N_1 x^{\alpha_1}\int x^{-\alpha_1-1}X\,dx + N_2 x^{\alpha_2}\int x^{-\alpha_2-1}X\,dx + \cdots\cdots$$
$$\cdots\cdots + N_n x^{\alpha_n}\int x^{-\alpha_n-1}X\,dx.$$

If a term $\frac{N}{(D'-\alpha)^r}$ be one of the partial fractions of $\frac{1}{f(D')}X$, then the operator $\frac{1}{D'-\alpha}$ is to be applied r times to X. Thus

$$\frac{1}{(D'-\alpha)^2}X = \frac{1}{D'-\alpha}x^{\alpha}\int x^{-\alpha-1}X\,dx$$
$$= x^{\alpha}\int x^{-1}\int x^{-\alpha-1}X\,(dx)^2$$

and, in general, $\frac{1}{(D'-\alpha)^r}X = x^{\alpha}\int x^{-1}\int x^{-1}\int \ldots\ldots\int x^{-\alpha-1}X(dx)^r$.

6.4. A particular case.

Let us consider the case when $X = x^m$.

We know that $D'x^m = x\frac{d}{dx}x^m = x\cdot mx^{m-1} = mx^m$,

$$D'^2x^m = x\frac{d}{dx}(mx^m) = x\cdot m^2x^{m-1} = m^2x^m,$$

$$D'^3x^m = x\frac{d}{dx}(m^2x^m) = x\,m^2\cdot mx^{m-1} = m^3x^m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore $f(D')x^m = f(m)x^m$.

Operating both sides with $\dfrac{1}{f(D')}$, we obtain

$$\frac{1}{f(D')}\{f(D')x^m\}=\frac{1}{f(D')}\{f(m)x^m\}$$

or, $\quad x^m=f(m)\dfrac{1}{f(D')}x^m$, since $f(m)$ is a constant.

Therefore $\quad \dfrac{1}{f(D')}x^m=\dfrac{1}{f(m)}x^m$

This method fails, if $f(m)=0$, that is, if $(D'-m)$ be a factor of $f(D')$. In that case, let

$$f(D')=(D'-m)\,F(D').$$

Then the particular integral is

$$\frac{1}{D'-m}\,\frac{1}{F(D')}x^m=\frac{1}{F(m)}\,\frac{1}{D'-m}x^m$$

$$=\frac{1}{F(m)}x^m\int x^{-m-1}x^m\,dx=\frac{x^m\log x}{F(m)}.$$

If $(D'-m)^r$ be a factor of $f(D')$, r being a positive integer, then

$$f(D')=(D'-m)^r\,G(D'),\ \text{say}.$$

Therefore the corresponding particular integral will be given by

$$\frac{1}{G(m)}\,\frac{1}{(D'-m)^r}x^m=\frac{x^m(\log x)^r}{r!\,G(m)}.$$

6.5. Equations reducible to homogeneous linear form.

The equations of the form

$$(a+bx)^n\frac{d^ny}{dx^n}+P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots$$

$$\cdots\cdots+P_{n-1}(a+bx)\frac{dy}{dx}+P_ny=X,\qquad\ldots(1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants and X is either a constant or a function of x only, can easily be reduced to the homogeneous linear form and hence also to the form of linear equations with constant coefficients. For this purpose, we write $a+bx=z$, so that

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=b\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(b\frac{dy}{dz}\right) = \frac{d}{dz}\left(b\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = b^2\frac{d^2y}{dz^2},$$

. . . .

. . . .

$$\frac{d^ny}{dx^n} = b^n\frac{d^ny}{dz^n}.$$

Substituting these in (1), we get the reduced equation as

$$z^n\frac{d^ny}{dz^n} + \frac{P_1}{b}z^{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \frac{P_2}{b^2}z^{n-2}\frac{d^{n-2}y}{dz^{n-2}} + \cdots\cdots$$

$$\cdots\cdots + \frac{P_{n-1}}{b^{n-1}}z\frac{dy}{dz} + \frac{P_n}{b^n}y = \frac{1}{b^n}Z, \quad \ldots \quad (2)$$

where Z is a function of z into which X is transformed by the substitution $x = \dfrac{z-a}{b}$.

This is an equation of homogeneous form and can be easily solved.

If $y = G(z)$ be the solution of the equation (2), then

$y = G(a + bx)$ is the solution of the equation (1).

If e^t had been substituted for $(a + bx)$, the independent variable thus being changed to t from x, we would get a linear equation with constant coefficients.

6.6. Illustrative Examples.

Ex. 1. *Solve* : $\quad x^3\dfrac{d^3y}{dx^3} - x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 2y = x^3 + 3x.$

We first change the independent variable x to z by the substitution $x = e^z$, that is, $z = \log x$, so that $x\dfrac{d}{dx} \equiv \dfrac{d}{dz} \equiv D'$, say.

The equation is then reduced to

$$D'(D'-1)(D'-2) - D'(D'-1) + 2D' - 2]\, y = e^{3z} + 3e^z$$

or, $\quad (D'^3 - 4D'^2 + 5D' - 2)y = e^{3z} + 3e^z$

or, $\quad (D'-1)^2(D'-2)y = e^{3z} + 3e^z.$

Here the auxiliary equation $(m-1)^2(m-2) = 0$ has the roots 1, 1, 2.

Thus the complementary function is

$$(C_1 + C_2 z)e^z + C_3 e^{2z} = (C_1 + C_2\log x)\,x + C_3 x^2.$$

The particular integral is

$$\frac{1}{(D'-1)^2(D'-2)}(e^{3z}+3e^z)$$

$$=\frac{1}{(D'-1)^2(D'-2)}e^{3z}+3\frac{1}{(D'-1)^2(D'-2)}e^z$$

$$=\frac{1}{4}e^{3z}-3\frac{1}{(D'-1)^2}e^z$$

$$=\frac{1}{4}e^{3z}-3e^z\frac{1}{(D'+1-1)^2}1$$

$$=\frac{1}{4}e^{3z}-3e^z\frac{1}{D'^2}1$$

$$=\frac{1}{4}e^{3z}-3e^z\frac{z^2}{2}=\frac{1}{4}x^3-\frac{3}{2}x(\log x)^2.$$

Hence the complete solution is

$$y=(C_1+C_2\log x)x+C_3x^2+\frac{1}{4}x^3-\frac{3}{2}x(\log x)^2.$$

Ex. 2. *Solve* : $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$. [C.H.1991,1993]

Let us put $x=e^z$, that is, $z=\log x$, so that $x\frac{d}{dx}\equiv\frac{d}{dz}\equiv D'$, say.

Then the given equation becomes

$$\{D'(D'-1)(D'-2)+2D'(D'-1)+2\}y=10(e^z+e^{-z})$$

or, $(D'+1)(D'^2-2D'+2)y=10(e^z+e^{-z})$.

The roots of the auxiliary equation $(m+1)(m^2-2m+2)=0$ are $-1, 1\pm i$.

Thus the complementary function is

$$C_1e^{-z}+(C_2\cos z+C_3\sin z)e^z$$

$$=C_1x^{-1}+\{C_2\cos(\log x)+C_3\sin(\log x)\}x.$$

The particular integral is

$$\frac{1}{(D'+1)(D'^2-2D'+2)}10(e^z+e^{-z})$$

$$=\frac{1}{(D'+1)(D'^2-2D'+2)}10e^z+\frac{1}{(D'+1)(D'^2-2D'+2)}10e^{-z}$$

$$=5e^z+\frac{1}{D'+1}2e^{-z}=5e^z+e^{-z}\frac{1}{D'-1+1}2$$

$$=5e^z+e^{-z}2z=5x+2x^{-1}\log x.$$

Hence the complete solution is

$y = x\{C_2 \cos(\log x) + C_3 \sin(\log x) + 5\} + x^{-1}(C_1 + 2\log x).$

Ex. 3 *Solve* : $(x^2 D^2 - 3xD + 5)\, y = x^2 \sin(\log x)$, *where* $D \equiv \dfrac{d}{dx}$.

Let us put $x = e^z$, so that $z = \log x$.

Then the given equation reduces to

$$\{D'(D' - 1) - 3D' + 5\}\, y = e^{2z} \sin z, \text{ where } D' \equiv x\frac{d}{dx} \equiv \frac{d}{dz}$$

or, $(D'^2 - 4D' + 5)\, y = e^{2z} \sin z$.

The roots of the auxiliary equation $m^2 - 4m + 5 = 0$ are $2 \pm i$.

Thus the complementary function is

$$e^{2z}(A\cos z + B \sin z) = x^2\{A\cos(\log x) + B\sin(\log x)\}.$$

The particular integral is

$$\frac{1}{D'^2 - 4D' + 5} e^{2z} \sin z = e^{2z} \frac{1}{(D' + 2)^2 - 4(D' + 2) + 5} \sin z$$

$$= e^{2z} \frac{1}{D'^2 + 1} \sin z$$

$$= e^{2z}\left(-\frac{z}{2}\cos z\right)$$

$$= -\frac{1}{2} x^2 \log x \cos(\log x).$$

Hence the complete solution is

$y = x^2\{A\cos(\log x) + B\sin(\log x) - \frac{1}{2}\log x \cos(\log x)\}.$

CHAPTER VII

EXACT DIFFERENTIAL EQUATIONS AND SOME SPECIAL FORMS OF EQUATIONS

7.1. Exact differential equations and criterion for exactness.

A differential equation of the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots\ldots, \frac{dy}{dx}, y\right) = X,$$

where X is a constant or a function of x only, is said to be *exact*, if it can be obtained by differentiating directly and without any further process an equation of the next lower order of the form

$$g\left(\frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots\ldots, \frac{dy}{dx}, y\right) = \int X\,dx + C,$$

C being a constant. The lower order equation is said to be the *first integral* of the higher order equation.

Let us consider the differential equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots\cdots + P_n y = X, \qquad \ldots \quad (1)$$

where $P_0, P_1, \ldots\ldots, P_n$ and X are constants or functions of x only.

Now we shall find a condition, which the coefficients of the differential equation must satisfy in order that it is exact. Let the successive derivatives of P's be denoted by dashes, that is, by

$$P', P'', \ldots\ldots, P^{(n)}.$$

On direct integration, we get

$$\int P_n y\,dx = \int P_n y\,dx,$$

$$\int P_{n-1} \frac{dy}{dx} dx = P_{n-1} y - \int P'_{n-1} y\,dx,$$

$$\int P_{n-2} \frac{d^2 y}{dx^2} dx = P_{n-2} \frac{dy}{dx} - \int P'_{n-2} \frac{dy}{dx} dx$$

$$= P_{n-2} \frac{dy}{dx} - P'_{n-2} y + \int P''_{n-2}\, y\,dx,$$

$$\int P_{n-3} \frac{d^3 y}{dx^3} dx = P_{n-3} \frac{d^2 y}{dx^2} - P'_{n-3} \frac{dy}{dx} + P''_{n-3}\, y - \int P'''_{n-3}\, y\,dx$$

and so on.

Thus we see that term by term integration of the differential equation gives

$$\int\left\{P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots + (-1)^n P_0^{(n)}\right\} y\,dx$$

$$+ (P_{n-1} - P'_{n-2} + P''_{n-3} - \cdots) y + (P_{n-2} - P'_{n-3} + \cdots)\frac{dy}{dx}$$

$$+ \left(P_{n-3} - P'_{n-4} + \cdots\right)\frac{d^2y}{dx^2} + \cdots + P_0\frac{d^{n-1}y}{dx^{n-1}} = \int X\,dx + C. \quad \ldots (2)$$

Now, the condition for exactness of the equation (1) will evidently be that there will be no term remaining which involves an integral of y in (2). Thus the required condition is

$$P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots + (-1)^n P_0^{(n)} = 0. \quad \ldots (3)$$

When this condition is satisfied, the first integral of (1) is

$$P_0\frac{d^{n-1}y}{dx^{n-1}} + (P_1 - P'_0)\frac{d^{n-2}y}{dx^{n-2}} + (P_2 - P'_1 + P''_0)\frac{d^{n-3}y}{dx^{n-3}} + \cdots$$

$$\cdots + \left\{P_{n-1} - P'_{n-2} + \cdots + (-1)^{n-1} P_0^{(n-1)}\right\} y$$

$$= \int X\,dx + C. \quad \ldots (4)$$

Note 1. Sometimes an equation, which is not exact, can be made exact by multiplying the equation by some suitable function of x known as integrating factor. If the coefficients be polynomial functions in x, then the integrating factor will be of the form x^m and m will be determined from the condition of exactness. In case the coefficients are trigonometric functions, then the integrating factor also will be trigonometric function and will be found by trial method.

Note 2. Sometimes we solve non-linear exact equations by a trial method, in which the terms of the equation are grouped in such a way that they become perfect differentials and then their integrals are written directly. We shall illustrate this process by examples.

7.2. Illustrative Examples.

Ex. 1. *Solve* : $(1+x^2)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=0$.

Here $P_0=1+x^2$, $P_1=3x$ and $P_2=1$.

Now $P_2-P_1'+P_0''=1-3+2=0$.

Hence the given equation is exact and its first integral is

$$P_0\frac{dy}{dx}+(P_1-P_0')\,y=C_1,\ C_1 \text{ being a constant,}$$

that is, $$(1+x^2)\frac{dy}{dx}+(3x-2x)\,y=C_1$$

or, $$\frac{dy}{dx}+\frac{x}{1+x^2}y=\frac{C_1}{1+x^2},$$

which is a linear equation of first order.

The integrating factor is

$$e^{\int\frac{x}{1+x^2}dx}=e^{\frac{1}{2}\log(1+x^2)}=\sqrt{1+x^2}.$$

Hence the general solution is

$$y\sqrt{1+x^2}=\int\frac{C_1}{1+x^2}\sqrt{1+x^2}\,dx$$

$$=C_1\int\frac{dx}{\sqrt{1+x^2}}=C_1\log(x+\sqrt{1+x^2})+C_2,$$

C_1 and C_2 being arbitrary constants.

Ex. 2. *Solve* : $(x^3-4x)\frac{d^3y}{dx^3}+(9x^2-12)\frac{d^2y}{dx^2}+18x\frac{dy}{dx}+6y=0$.

Here $P_0=x^3-4x$, $P_1=9x^2-12$, $P_2=18x$, $P_3=6$.

We have $P_3-P_2'+P_1''-P_0'''=6-18+18-6=0$.

Hence the given equation is exact and its first integral is

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\,y=C_1,$$

that is, $$(x^3-4x)\frac{d^2y}{dx^2}+(9x^2-12-3x^2+4)\frac{dy}{dx}+(18x-18x+6x)\,y=C_1$$

or, $$(x^3-4x)\frac{d^2y}{dx^2}+(6x^2-8)\frac{dy}{dx}+6xy=C_1. \quad \ldots (1)$$

Here, again, $P_0=x^3-4x$, $P_1=6x^2-8$, $P_2=6x$.

We have $P_2 - P_1' + P_0'' = 6x - 12x + 6x = 0$.

Hence the equation (1) is exact and its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = C_1 x + C_2,$$

that is, $(x^3 - 4x)\frac{dy}{dx} + (6x^2 - 8 - 3x^2 + 4)\, y = C_1 x + C_2$

or, $(x^3 - 4x)\frac{dy}{dx} + (3x^2 - 4)\, y = C_1 x + C_2$. ... (2)

Here, again, $P_0 = x^3 - 4x$, $P_1 = 3x^2 - 4$.

We have $P_1 - P_0' = 3x^2 - 4 - 3x^2 + 4 = 0$.

Hence the equation (2) is also exact and its first integral is

$$P_0\, y = C_1 \frac{x^2}{2} + C_2\, x + C_3$$

or, $(x^3 - 4x)\, y = C_1 \frac{x^2}{2} + C_2\, x + C_3$, C_1, C_2, C_3 being arbitrary constants.

Ex. 3. *Solve* : $\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = 0$. [*C. H. 1994*]

Here $P_0 = \sin x$, $P_1 = -\cos x$ and $P_2 = 2 \sin x$.

Now $P_2 - P_1' + P_0'' = 2\sin x - \sin x - \sin x = 0$.

Hence the equation is exact and its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = C_1$$

or, $\sin x \frac{dy}{dx} - 2y \cos x = C_1$

or, $\frac{dy}{dx} - 2\cot x \, . \, y = C_1 \operatorname{cosec} x$, which is a linear equation of first order and whose integrating factor is $\operatorname{cosec}^2 x$.

Hence its solution is $y \operatorname{cosec}^2 x = C_1 \int \operatorname{cosec}^3 x \, dx$

or, $y \operatorname{cosec}^2 x = \frac{C_1}{2}\left(\log \tan \frac{x}{2} - \operatorname{cosec} x \cot x \right) + C_2$

or, $y = \frac{C_1}{2} \sin^2 x \log \tan \frac{x}{2} - \frac{C_1}{2} \cos x + C_2 \sin^2 x$.

CHAPTER VIII

LINEAR EQUATIONS OF SECOND ORDER

8.1. Complete solution in terms of known integral.

Theorem. *If an integral included in the complementary function of a linear equation of second order be known, then the complete solution can be expressed in terms of that known integral.* 2016.

We consider a linear equation of second order in the standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \qquad \ldots \qquad (1)$$

where P, Q and X are functions of x.

If $X = 0$, then the equation is said to be the *reduced equation* of (1).

Let $y = u$, a function of x, be a known integral in the complementary function of (1). We shall now try to determine the complete solution of (1) in terms of u. For that purpose, let us assume the complete solution as $y = uv$, where v is also a function of x.

Then we have $y_1 = u_1 v + uv_1$ and $y_2 = u_2 v + 2u_1 v_1 + uv_2$,

where the suffixes denote the order of differentiation with respect to x.

Substituting these values of y, y_1 and y_2 in (1), we get

$$uv_2 + (2u_1 + Pu)\,v_1 + (u_2 + Pu_1 + Qu)\,v = X. \qquad \ldots \qquad (2)$$

Now, since $y = u$ is a solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

the coefficient of v in (2) vanishes and we have

$$v_2 + \left(2\frac{u_1}{u} + P\right)v_1 = \frac{X}{u},$$

which is a linear equation in v_1 and hence v_1 can be determined which will include one constant.

Then we get $v = \int v_1\,dx + C$.

Thus the complete primitive is given by

$$y = uv = u\int v_1\,dx + Cu.$$

8.2. A solution found by inspection.

Since the complete solution of a linear equation of second order can be found, if one integral in its complementary function be known, we shall try here to determine, by inspection, such an integral of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0. \qquad \ldots \quad (1)$$

(*a*) Let $y = e^{mx}$ be a solution of the equation (1).

Then we have $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2 e^{mx}$.

Putting these in the equation, we get

$$m^2 + Pm + Q = 0, \text{ since } e^{mx} \neq 0.$$

Hence (*i*) e^x will be a solution ($m = 1$), if $1 + P + Q = 0$,

(*ii*) e^{-x} will be a solution ($m = -1$), if $1 - P + Q = 0$,

(*iii*) e^{ax} will be a solution ($m = a$), if $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$.

(*b*) Let $y = x^m$ be a solution of the equation (1).

Then we have $\frac{dy}{dx} = mx^{m-1}$ and $\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$.

Putting these in the equation, we get

$$m(m-1) + Pmx + Qx^2 = 0.$$

Hence (*i*) $y = x$ is a solution ($m = 1$), if $P + Qx = 0$,

(*ii*) $y = x^2$ is a solution ($m = 2$), if $2 + 2Px + Qx^2 = 0$.

8.3. Linear dependence : Wronskian.

The functions $f_1(x), f_2(x), \ldots\ldots, f_n(x)$ are said to be *linearly dependent* on an interval T, if there exist constants $c_1, c_2, \ldots\ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots\cdots + c_n f_n(x) = 0, \qquad \ldots \quad (1)$$

for all $x \in T$.

Functions that are not linearly dependent are called *linearly independent*. Thus the functions $f_1, f_2, \ldots\ldots, f_n$ of x are linearly independent on T, if the only constants, that satisfy

$$c_1 f_1(x) + c_2 f_2(x) + \cdots\cdots + c_n f_n(x) = 0,$$

are the constants $c_1 = c_2 = \cdots\cdots = c_n = 0$, for all $x \in T$.

The immediate consequence of linear dependence of a given set of functions $f_1, f_2, \ldots\ldots, f_n$, is that, assuming $c_r \neq 0$ in (1), we have

$$f_r(x) = \beta_1 f_1(x) + \cdots\cdots + \beta_{r-1} f_{r-1}(x) + \beta_{r+1} f_{r+1}(x) + \cdots \cdots + \beta_n f_n(x),$$

in which $\beta_i = -\dfrac{c_i}{c_r}$.

This shows that the function $f_r(x)$ can be expressed as a linear combination of the remaining $(n-1)$ functions of the given set.

The *Wronskian* of two differentiable functions $f_1(x)$ and $f_2(x)$ on an interval T is defined by the determinant

$$W(f_1, f_2 : x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

where the dash denotes differentiation with respect to the variable.

The definition can be extended for the case of more than two functions, in which there will be higher order derivatives of the functions. A Wronskian with n functions will contain derivatives of functions of all orders up to $(n-1)$, each row below the first will contain elements that are the derivatives of the corresponding elements in the row above them.

It can be easily verified that if W be the Wronskian of the functions

$$1, x, x^2, \ldots\ldots, x^{n-1} \text{ for } n > 1,$$

then $W = 0!\, 1!\, 2! \ldots\ldots (n-1)!$.

We shall now obtain a sufficient condition that n functions will be linearly independent over an interval T.

Let us assume that each of the functions $f_1, f_2, \ldots\ldots, f_n$ is differentiable at least $(n-1)$ times in the interval T. Then, from the equation $c_1 f_1 + c_2 f_2 + \cdots\cdots + c_n f_n = 0$, we get, by successive differentiation,

$$c_1 f_1' + c_2 f_2' + \cdots\cdots + c_n f_n' = 0,$$

$$c_1 f_1'' + c_2 f_2'' + \cdots\cdots + c_n f_n'' = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_1 f_1^{\,n-1} + c_2 f_2^{\,n-1} + \cdots\cdots + c_n f_n^{\,n-1} = 0.$$

Now, if we consider these n linear equations as a system of equations in $c_1, c_2, \ldots\ldots, c_n$, and if the determinant of the system does not vanish, the system will have no solution except the one with each of the c's equal to zero. Thus, if the Wronskian $W(f_1, f_2, \ldots\ldots, f_n : x) \neq 0$, then the functions $f_1, f_2, \ldots\ldots, f_n$ are linearly independent. Hence the non-vanishing of the Wronskian is a sufficient condition that the functions are linearly independent.

The non-vanishing of the Wronskian on an interval however, is not a necessary condition for linear independence.

It can be shown easily that the functions

(*i*) e^x, e^{2x}, e^{3x}; (*ii*) e^x, $\cos x$, $\sin x$ and (*iii*) 1, $\sin x$, $\cos x$ are linearly independent, while the functions

(*iv*) $1, \sin^2 x, \cos^2 x$; (*v*) $\sin 3x$, $\sin x$, $\sin^3 x$ and (*vi*) $x^2 - x + 1,\ x^2 - 1,\ 3x^2 - x - 1$ are linearly dependent.

8.4. Relation between the two integrals.

Let u and v be the two independent integrals of the linear equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0, \qquad \ldots (1)$$

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so that

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

and

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0.$$

Eliminating Q from these two equations, we get

$$\left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right) + P\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) = 0. \qquad \ldots (2)$$

Now, if we put $w = u\dfrac{dv}{dx} - v\dfrac{du}{dx}$, $(\neq 0)$, then $$\frac{dw}{dx} = u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2};$$

so that (2) becomes $$\frac{dw}{dx} + Pw = 0$$

or, $$w = Ae^{-\int P\,dx}$$

Thus we see that any two particular integrals u and v of the equation (1) are connected by the relation

$$u\frac{dv}{dx} - v\frac{du}{dx} = Ae^{-\int P\,dx}, \quad A \text{ being a constant.}$$

Here $w = u\dfrac{dv}{dx} - v\dfrac{du}{dx} = \text{Wronskian } W(u, v) = Ae^{-\int P\,dx}$

8.5. Solution found by operational factors.

If the left hand side of the linear equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

be put in the form $f(D)\,y$, then it is sometimes convenient to factorise $f(D)$ in the form $f_2(D)\,f_1(D)$, such that when $f_1(D)$ operates upon y and then $f_2(D)$ operates upon the result of this operation, the same result is obtained as when $f(D)$ operates upon y.

This is symbolically expressed as

$$f(D)\,y = f_2(D)\,\{f_1(D)\,y\}$$
$$= f_2(D)\,f_1(D)\,y,$$

in which the operations are made from right to left. As the factors of $f(D)$ are not commutative since they contain the variables, the factors must be put in right order. We shall make the method of procedure clear with illustrative examples.

8.6. Illustrative Examples.

Ex. 1. *Solve :* $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\,y = 0$, *in terms of known integral.* [C. H. 1994]

Dividing both sides of the given equation by x, we get the equation in the standard form as

$$\frac{d^2y}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = 0,$$

so that here $P = -\left(2 - \frac{1}{x}\right),\ Q = 1 - \frac{1}{x}$.

By inspection, we say that e^x is a solution of the equation, since $1 + P + Q = 0$ here.

Now, for a complete solution, we put $y = ve^x$,

so that $\frac{dy}{dx} = ve^x + e^x \frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = ve^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}$.

Putting these in the equation, we get

$$x \frac{d^2v}{dx^2} + \frac{dv}{dx} = 0$$

or,

$$\frac{d}{dx}\left(x \frac{dv}{dx}\right) = 0.$$

Integrating, we get $x \frac{dv}{dx} = C_1$, where C_1 is a constant.

This gives

$$\frac{dv}{dx} = \frac{C_1}{x}.$$

Integrating once again, we get $v = C_1 \log x + C_2$, C_2 being a constant.

Hence the complete solution is

$$y = ve^x = e^x (C_1 \log x + C_2).$$

Ex. 2. *Solve* : $(1 - x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x(1 - x^2)^{\frac{3}{2}}$, *in terms of known integral.* [*C. H. 1988*]

Writing the equation in the standard form, we get

$$\frac{d^2y}{dx^2} + \frac{x}{1 - x^2} \frac{dy}{dx} - \frac{1}{1 - x^2} y = x(1 - x^2)^{\frac{1}{2}},$$

so that here $P = \frac{x}{1 - x^2},\ Q = -\frac{1}{1 - x^2}$ and $P + Qx = 0$.

Thus $y = x$ is a part of the complementary function.

Now, for a complete solution, we put

$y = vx$, so that $\frac{dy}{dx} = x \frac{dv}{dx} + v$ and $\frac{d^2y}{dx^2} = x \frac{d^2v}{dx^2} + 2 \frac{dv}{dx}$.

Putting these in the equation, we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} + \frac{x}{1 - x^2}\right) \frac{dv}{dx} = (1 - x^2)^{\frac{1}{2}}.$$

Substituting p for $\frac{dv}{dx}$, this becomes

$$\frac{dp}{dx}+\left(\frac{2}{x}+\frac{x}{1-x^2}\right)p=(1-x^2)^{\frac{1}{2}}.$$

This is a linear equation of first order in p, whose integrating factor is

$$\frac{x^2}{\sqrt{1-x^2}}.$$

Thus the solution is

$$p\,\frac{x^2}{\sqrt{1-x^2}}=\frac{1}{3}x^3+C_1$$

or,

$$\frac{dv}{dx}=p=\frac{1}{3}x\sqrt{1-x^2}+\frac{C_1}{x^2}\sqrt{1-x^2}.$$

Integrating, we get

$$v=-\frac{1}{9}(1-x^2)^{\frac{3}{2}}-C_1\left(\sin^{-1}x+\frac{1}{x}\sqrt{1-x^2}\right)+C_2.$$

Hence the complete integral is

$$y=vx=-\frac{1}{9}x(1-x^2)^{\frac{3}{2}}-C_1\left(x\sin^{-1}x+\sqrt{1-x^2}\right)+C_2x,$$

C_1 and C_2 being arbitrary constants.

8.7. Reduction to normal form by removing the first derivative.

We have seen that a linear equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \qquad \ldots \quad (1)$$

can only be solved when an integral solution belonging to the complementary function is known. But it is not always possible to find such an integral solution. We give below a method for the solution of an equation of the form (1) when no such integral solution can be found.

For that purpose, we assume $y = uv$,

where both u and v are functions of x and none of u and v is a part of the complementary function of (1).

Now $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ and $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2}$.

Putting these in (1), we get

$$u\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = X$$

or, $$u\frac{d^2v}{dx^2} + u\left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = X. \qquad \ldots \quad (2)$$

To remove the term of the first derivative of v, we choose u in such a manner that

$$P + \frac{2}{u}\frac{du}{dx} = 0 \qquad \ldots \quad (3)$$

or, $$u = e^{-\frac{1}{2}\int P\,dx} \qquad \ldots \quad (4)$$

Now the equation (2) gives

$$\frac{d^2v}{dx^2} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = \frac{X}{u}. \quad \ldots \quad (5)$$

Also we have, from (3), $\frac{du}{dx} = -\frac{1}{2}Pu$

and hence $$\frac{d^2u}{dx^2} = -\frac{1}{2}\left(P\frac{du}{dx} + u\frac{dP}{dx}\right)$$

$$= -\frac{1}{2}\left\{P\left(-\frac{1}{2}Pu\right) + u\frac{dP}{dx}\right\} = \frac{P^2}{4}u - \frac{u}{2}\frac{dP}{dx}.$$

Putting these in (5), we get the equation reduced to the *normal form* as

$$\frac{d^2v}{dx^2} + Iv = S, \quad \ldots \quad (6)$$

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ and $S = Xe^{\frac{1}{2}\int P\,dx}$

If the value of I be a constant or a constant divided by x^2, then the equation (6) can readily be solved.

8.8. Change of independent variable.

Sometimes an equation is transformed to an integrable form by changing the independent variable by a suitable substitution.

Let the linear equation of second order in its standard form be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \quad \ldots \quad (1)$$

where P, Q and X are functions of x.

Let us change the independent variable x to z, so that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\frac{dz}{dx}\right)\frac{dz}{dx} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}.$$

Putting these in the equation (1), we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = X$$

$$\text{or, } \frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{X}{\left(\frac{dz}{dx}\right)^2}$$

$$\text{or, } \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = X_1, \qquad \ldots \quad (2)$$

$$\text{where } P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \; Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } X_1 = \frac{X}{\left(\frac{dz}{dx}\right)^2}. \qquad \ldots \quad (3)$$

P_1, Q_1, X_1 are functions of x as shown above but can be expressed as functions of z, if a relation be given between z and x.

Now, if we choose z in such a way that P_1 vanishes, that is,

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0,$$

that is, $$z = \int e^{-\int P\,dx}\,dx,$$

then (2) changes to $\dfrac{d^2y}{dz^2} + Q_1\,y = X_1$.

This is solvable, if Q_1 comes out to be a constant or a constant divided by z^2.

If, again, we choose z in such a way that

$Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ is a constant, say a^2, then

$$a\frac{dz}{dx} = \sqrt{Q}$$

or, $$az = \int \sqrt{Q}\,dx.$$

With this substitution, (2) becomes

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + a^2 y = X_1 .$$

This equation can easily be solved, if P_1 also comes out to be a constant.

Note. Sometimes it is possible to make both the choices to get the solution of the given equation.

8.9. Illustrative Examples.

Ex. 1. *Solve :* $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right) y = 0$, *by reducing to normal form.* *[N. B. H. 1999 2004]*

Comparing this equation with the linear equation of second order

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X ,$$

we get $$P = -\frac{2}{x}, \; Q = a^2 + \frac{2}{x^2} \text{ and } X = 0 .$$

Now, if we take

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int\left(-\frac{2}{x}\right)dx} = e^{\int \frac{1}{x}dx} = x ,$$

then, on substitution of $y = uv = vx$, the given equation becomes

$$\frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right) v = 0$$

or, $$\frac{d^2v}{dx^2} + \left(a^2 + \frac{2}{x^2} - \frac{1}{x^2} - \frac{1}{2}\frac{2}{x^2}\right) v = 0$$

or, $$(D^2 + a^2) v = 0 , \text{ where } D \equiv \frac{d}{dx} .$$

This gives $v = C_1 \cos ax + C_2 \sin ax$.

Hence the complete solution of the given equation is

$$y = uv = (C_1 \cos ax + C_2 \sin ax) x .$$

8.10. Method of variation of parameters.

This method, as explained earlier for equations of first order, is used to find the complete primitive of a linear differential equation, when its complementary function is known. We shall explain the method for a linear differential equation of second order, but it can be extended to linear equations of any order, though in that case it requires too much labour to solve a number of simultaneous equations.

The complete solution is obtained by varying the parameters of the complementary function.

Let us consider the general linear equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \qquad \ldots \quad (1)$$

where P, Q and X are functions of x.

Let $\quad y = Au + Bv \qquad \ldots \quad (2)$

be the complementary function of the equation (1), where A and B are constants and u and v are functions of x and are independent solutions of the corresponding homogeneous equation. Thus

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \text{ and } \frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0. \qquad \ldots \quad (3)$$

Let us assume that $y = \phi u + \psi v \qquad \ldots \quad (4)$

is the complete primitive of (1), where we take ϕ and ψ in place of A and B and they are no longer constants but functions of x, to be so chosen that (4) will satisfy (1).

Differentiating (4) with respect to x, we get

$$\frac{dy}{dx} = \phi\frac{du}{dx} + \psi\frac{dv}{dx} + u\frac{d\phi}{dx} + v\frac{d\psi}{dx}.$$

Let us choose ϕ and ψ such that

$$u\frac{d\phi}{dx} + v\frac{d\psi}{dx} = 0, \qquad \ldots \quad (5)$$

so that $$\frac{dy}{dx} = \phi\frac{du}{dx} + \psi\frac{dv}{dx}.$$

Differentiating this once again, we have

$$\frac{d^2y}{dx^2} = \phi\frac{d^2u}{dx^2} + \psi\frac{d^2v}{dx^2} + \frac{d\phi}{dx}\frac{du}{dx} + \frac{d\psi}{dx}\frac{dv}{dx}.$$

Substituting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\phi\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + \psi\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right) + \frac{d\phi}{dx}\frac{du}{dx} + \frac{d\psi}{dx}\frac{dv}{dx} = X$$

or, $$\frac{d\phi}{dx}\frac{du}{dx} + \frac{d\psi}{dx}\frac{dv}{dx} = X, \qquad \ldots \quad (6)$$

by virtue of (3).

Solving for $\frac{d\phi}{dx}$ and $\frac{d\psi}{dx}$ from (5) and (6), we get

$$\frac{d\phi}{dx} = \frac{vX}{v\frac{du}{dx} - u\frac{dv}{dx}} \text{ and } \frac{d\psi}{dx} = -\frac{uX}{v\frac{du}{dx} - u\frac{dv}{dx}}.$$

Integrating, we get

$$\phi = C_1 + \int \frac{vX\,dx}{v\frac{du}{dx} - u\frac{dv}{dx}} \text{ and } \psi = C_2 - \int \frac{uX\,dx}{v\frac{du}{dx} - u\frac{dv}{dx}}.$$

Substituting these values of ϕ and ψ in (4), we get the complete solution of the equation (1), C_1 and C_2 being arbitrary constants.

Cor. For an equation of third order $\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = S$, ... (1)

let the complementary function be $Au + Bv + Cw$, where u, v, w are linearly independent functions of x and A, B, C are constants.

Let $$y = u\phi + v\psi + w\chi \quad \text{... (2)}$$

be the complete primitive of (1), where ϕ, ψ, χ are taken in place of A, B, C and they are no longer constants but functions of x. Then we have

$$\frac{dy}{dx} = \frac{du}{dx}\phi + \frac{dv}{dx}\psi + \frac{dw}{dx}\chi, \quad \text{... (3)}$$

provided that $$u\frac{d\phi}{dx} + v\frac{d\psi}{dx} + w\frac{d\chi}{dx} = 0. \quad \text{... (4)}$$

Hence $$\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2}\phi + \frac{d^2v}{dx^2}\psi + \frac{d^2w}{dx^2}\chi, \quad \text{... (5)}$$

provided that $$\frac{du}{dx}\frac{d\phi}{dx} + \frac{dv}{dx}\frac{d\psi}{dx} + \frac{dw}{dx}\frac{d\chi}{dx} = 0. \quad \text{... (6)}$$

Then $$\frac{d^3y}{dx^3} = \frac{d^3u}{dx^3}\phi + \frac{d^3v}{dx^3}\psi + \frac{d^3w}{dx^3}\chi$$

$$+ \frac{d^2u}{dx^2}\frac{d\phi}{dx} + \frac{d^2v}{dx^2}\frac{d\psi}{dx} + \frac{d^2w}{dx^2}\frac{d\chi}{dx}. \quad \text{... (7)}$$

By substitution in (1), we have $\frac{d^2u}{dx^2}\frac{d\phi}{dx} + \frac{d^2v}{dx^2}\frac{d\psi}{dx} + \frac{d^2w}{dx^2}\frac{d\chi}{dx} = S.$

... (8)

Then $\frac{d\phi}{dx}$, $\frac{d\psi}{dx}$ and $\frac{d\chi}{dx}$ are found from the three equations (4), (6) and (8).

8.11. Illustrative Examples.

Ex. 1. *Solve, by the method of variation of parameters, the equation*

$$\frac{d^2y}{dx^2} + a^2y = \sec ax .$$ [C. H. 1995, 2001]

The complementary function of the equation is

$$A \cos ax + B \sin ax,$$

in which A and B are constants.

Here $\cos ax$ and $\sin ax$ are independent solutions of the corresponding homogeneous equation, since their Wronskian

$$\begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$
$$= a\cos^2 ax + a\sin^2 ax$$
$$= a \neq 0 .$$

Now assume ϕ and ψ to be functions of x in place of A and B, in such a way that the given equation is satisfied completely by

$$y = \phi \cos ax + \psi \sin ax .$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\phi a \sin ax + \psi a \cos ax + \cos ax \frac{d\phi}{dx} + \sin ax \frac{d\psi}{dx} .$$

We choose ϕ and ψ such that

$$\cos ax \frac{d\phi}{dx} + \sin ax \frac{d\psi}{dx} = 0 . \quad \ldots \quad (1)$$

Therefore $\frac{dy}{dx} = -\phi a \sin ax + \psi a \cos ax$

and $$\frac{d^2y}{dx^2} = -\phi a^2 \cos ax - \psi a^2 \sin ax - a\sin ax \frac{d\phi}{dx} + a\cos ax \frac{d\psi}{dx} .$$

Putting these values of y and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$-a \sin ax \frac{d\phi}{dx} + a\cos ax \frac{d\psi}{dx} = \sec ax . \quad \ldots \quad (2)$$

Solving (1) and (2), we get

$$a\frac{d\phi}{dx} = -\tan ax \quad \text{and} \quad a\frac{d\psi}{dx} = 1 .$$

Integrating, we get

$$\phi = \frac{1}{a^2}\log\cos ax + C_1 \quad \text{and} \quad \psi = \frac{x}{a} + C_2 ,$$

where C_1 and C_2 are arbitrary constants.

Hence the complete solution of the equation is

$$y = \phi\cos ax + \psi\sin ax$$

$$= C_1\cos ax + C_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log\cos ax .$$

Ex. 2. *Solve, by the method of variation of parameters, the equation*

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x .$$ [*V. H. 1991; B. H. 1999; N. B. H. 2006*]

The complementary function of the equation is

$A\cos 2x + B\sin 2x$, A and B being constants.

Here $\cos 2x$ and $\sin 2x$ are independent solutions, since their Wronskian is not zero.

Assume the complete solution of the equation to be

$y = \phi\cos 2x + \psi\sin 2x$, where ϕ and ψ are functions of x.

Now $$\frac{dy}{dx} = -2\phi\sin 2x + 2\psi\cos 2x + \cos 2x\frac{d\phi}{dx} + \sin 2x\frac{d\psi}{dx} .$$

Choose ϕ and ψ such that $\cos 2x\dfrac{d\phi}{dx} + \sin 2x\dfrac{d\psi}{dx} = 0 .$ ... (1)

Therefore $$\frac{dy}{dx} = -2\phi\sin 2x + 2\psi\cos 2x .$$

Also $$\frac{d^2y}{dx^2} = -4\phi\cos 2x - 4\psi\sin 2x - 2\sin 2x\frac{d\phi}{dx} + 2\cos 2x\frac{d\psi}{dx} .$$

Putting these values of y and $\dfrac{d^2y}{dx^2}$ in the given equation, we get

$$\cos 2x\frac{d\psi}{dx} - \sin 2x\frac{d\phi}{dx} = 2\tan 2x . \quad \ldots (2)$$

Solving (1) and (2) for $\dfrac{d\phi}{dx}$ and $\dfrac{d\psi}{dx}$, we get

$$\frac{d\phi}{dx} = -\frac{2\sin^2 2x}{\cos 2x} \quad \text{and} \quad \frac{d\psi}{dx} = 2\sin 2x .$$

Integrating, we get

$$\phi = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx = -2\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= \sin 2x - \log(\sec 2x + \tan 2x) + C_1$$

and $$\psi = 2\int \sin 2x\,dx = C_2 - \cos 2x,$$

where C_1 and C_2 are arbitrary constants.

Hence the complete solution of the equation is

$$y = \phi\cos 2x + \psi\sin 2x$$

$$= \sin 2x - \log(\sec 2x + \tan 2x) + C_1 \quad \cos 2x$$

$$+ (C_2 - \cos 2x)\sin 2x$$

$$= C_1\cos 2x + C_2\sin 2x - \cos 2x\log(\sec 2x + \tan 2x).$$

CHAPTER IX

SIMULTANEOUS LINEAR EQUATIONS

9.1. Introduction.

So far we have considered only those differential equations which contain two variables, one independent and the other dependent. Now, in this chapter, we consider the methods of solutions of differential equations involving more than two variables. The simplest form of such equations is that, in which the number of independent variables is one. The number of equations, which will connect these variables, will be equal to the number of dependent variables. We shall consider here ordinary equations with one independent variable and two dependent variables.

9.2. Simultaneous linear equations with constant coefficients.

First Method :

Let x, y be the dependent variables and t be the independent variable. The equations will involve derivatives of x and y with respect to t. Let us denote the operator $\frac{d}{dt}$ by the symbol D. Then the simultaneous linear equations, to be solved, will be of the form

$$f_1(D)\,x + f_2(D)\,y = T_1 \qquad \ldots \quad (1)$$

and

$$\phi_1(D)\,x + \phi_2(D)\,y = T_2, \qquad \ldots \quad (2)$$

where $f_1(D)$, $f_2(D)$, $\phi_1(D)$, $\phi_2(D)$ are all rational functions of D with constant coefficients and T_1, T_2 are functions of t, the independent variable.

To eliminate y, we operate (1) with $\phi_2(D)$ and (2) with $f_2(D)$. Then these equations become

$$\phi_2(D) f_1(D)\,x + \phi_2(D) f_2(D)\,y = \phi_2(D)\,T_1 \qquad \ldots \quad (3)$$

and

$$f_2(D)\,\phi_1(D)\,x + f_2(D)\,\phi_2(D)\,y = f_2(D)\,T_2. \qquad \ldots \quad (4)$$

Now, since $f_2(D)$ and $\phi_2(D)$ are rational functions of D with constant coefficients, we have

$$\phi_2(D) f_2(D)\,y = f_2(D)\,\phi_2(D)\,y.$$

Hence, subtracting (4) from (3), we get

$$\{\phi_2(D) f_1(D) - f_2(D)\phi_1(D)\} x = \phi_2(D) T_1 - f_2(D) T_2,$$

which is of the form $F(D)x = T(t)$. ... (5)

This equation (5), being a linear differential equation, can be solved to find x as a function of t. Now the value of y can be obtained as a function of t by substituting the value of x in either of the two equations. If, however, y be determined by an independent elimination, as the case of x, the values of x and y so obtained will have to be substituted in equation (1) or (2) and the arbitrary constants are adjusted, so that the equations may be satisfied.

Note that the number of arbitrary constants in the complete solution of (1) and (2) will be equal to the degree of D in the polynomial $F(D)$ of (5).

Second method :

The two given equations connect t with the four quantities $x, y, \frac{dx}{dt}, \frac{dy}{dt}$. We differentiate them with respect to t and get four equations connecting $x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$. From these four equations, we eliminate three quantities $y, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$. In this way, an equation of the second order, in which x is the dependent variable and t is the independent variable, is obtained. This is solved to get x as a function of t. Then y is obtained by substituting this value of x in the equations already obtained.

Note that this method is applied only when the given simultaneous equations are of order one.

9.3. Illustrative Examples.

Ex. 1. *Solve :*

$$\frac{dx}{dt} - 7x + y = 0,$$

$$\frac{dy}{dt} - 2x - 5y = 0.$$

[B. H. 1991; K. H. 2003]

Using the symbol D for $\frac{d}{dt}$, the given equations can be written as

$$(D-7)x + y = 0, \quad \ldots (1)$$

$$-2x + (D-5)y = 0. \quad \ldots (2)$$

Eliminating y between (1) and (2), we get

$$\{(D-5)(D-7)+2\}x=0$$

or, $$(D^2-12D+37)x=0.$$

The auxiliary equation is $m^2-12m+37=0$, giving $m=6\pm i$.

Therefore $$x=e^{6t}(C_1\cos t+C_2\sin t). \quad \ldots \quad (3)$$

Then we have $$\frac{dx}{dt}=e^{6t}(-C_1\sin t+C_2\cos t)$$
$$+6e^{6t}(C_1\cos t+C_2\sin t). \quad \ldots \quad (4)$$

Substituting for x and $\frac{dx}{dt}$ in the first of the given equations, we get

$$y=C_1e^{6t}\cos t+C_2e^{6t}\sin t+C_1e^{6t}\sin t-C_2e^{6t}\cos t$$
$$=e^{6t}\{(C_1-C_2)\cos t+(C_1+C_2)\sin t\}. \quad \ldots \quad (5)$$

Hence the complete solution is given by (3) and (5).

Ex. 2. *Solve* : $$(4D+44)x+(9D+49)y=t,$$
$$(3D+34)x+(7D+38)y=e^t, \text{ where } D\equiv\frac{d}{dt}.$$

Eliminating x between the two given equations, we get

$$\{(4D+44)(7D+38)-(3D+34)(9D+49)\}y$$
$$=(4D+44)e^t-(3D+34)t$$

or, $$(D^2+7D+6)y=48e^t-34t-3.$$

This is a linear equation of second order.

The auxiliary equation is $m^2+7m+6=0$, whence $m=-6,-1$.

Therefore the complementary function is

$$C_1e^{-6t}+C_2e^{-t}.$$

The particular integral is

$$\frac{1}{D^2+7D+6}(48e^t-34t-3)$$
$$=\frac{48}{14}e^t-\frac{1}{6}\left\{1+\left(\frac{7}{6}D+\frac{1}{6}D^2\right)\right\}^{-1}(34t+3)$$
$$=\frac{24}{7}e^t-\frac{1}{6}\left(1-\frac{7}{6}D\right)(34t+3)$$
$$=\frac{24}{7}e^t-\frac{1}{6}\left(34t+3-\frac{7}{6}\cdot34\right)$$
$$=\frac{24}{7}e^t-\frac{17}{3}t+\frac{55}{9}.$$

Hence the general solution for y is

$$y = C_1 e^{-6t} + C_2 e^{-t} + \frac{24}{7} e^{t} - \frac{17}{3} t + \frac{55}{9}. \qquad \ldots \quad (1)$$

Therefore $$\frac{dy}{dt} = -6C_1 e^{-6t} - C_2 e^{-t} + \frac{24}{7} e^{t} - \frac{17}{3}. \qquad \ldots \quad (2)$$

Now, if we multiply the first equation by 3 and subtract the result from the second equation being multiplied by 4, we get

$$\frac{dy}{dt} + 4x + 5y = 4e^{t} - 3t. \qquad \ldots \quad (3)$$

Putting the values of y and $\frac{dy}{dt}$ from (1) and (2) in (3), we get

$$-6C_1 e^{-6t} - C_2 e^{-t} + \frac{24}{7} e^{t} - \frac{17}{3} + 5C_1 e^{-6t} + 5C_2 e^{-t}$$
$$+ \frac{120}{7} e^{t} - \frac{85}{3} t + \frac{275}{9} + 4x = 4e^{t} - 3t$$

or, $$4x = C_1 e^{-6t} - 4C_2 e^{-t} - \frac{116}{7} e^{t} + \frac{76}{3} t - \frac{224}{9}$$

or, $$x = \frac{1}{4} C_1 e^{-6t} - C_2 e^{-t} - \frac{29}{7} e^{t} + \frac{19}{3} t - \frac{56}{9}. \qquad \ldots \quad (4)$$

(1) and (4) constitute the solution of the given equations.

9.4. Simultaneous equations in a different form.

Let the equations be given in the form

$$P_1\,dx + Q_1\,dy + R_1\,dz = 0, \qquad \ldots \quad (1)$$

$$P_2\,dx + Q_2\,dy + R_2\,dz = 0, \qquad \ldots \quad (2)$$

where $P_1, Q_1, R_1, P_2, Q_2, R_2$ are functions of x, y and z.

Then, by cross-multiplication, we get

$$\frac{dx}{Q_1 R_2 - Q_2 R_1} = \frac{dy}{R_1 P_2 - R_2 P_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1},$$

which is of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, $\qquad \ldots \quad (3)$

where P, Q, R are functions of x, y, z.

Thus simultaneous equations of the forms (1) and (2) can always be put in the form (3).

The following methods are used for the solution of equations in the form (3) :

First method :

By equating two of the three members of (3), we may be able to get an equation in only two variables. The solution of this equation gives us one of the relations of the general solution of (3). This method may be repeated to get another relation with the help of two other members of (3). These two relations constitute the general solution. One relation so obtained may be used to simplify the other differential equations to put them in the integrable form.

Second method :

We may be able to find multipliers l, m, n and L, M, N such that one of the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}=\frac{L\,dx+M\,dy+N\,dz}{LP+MQ+NR}$$

can easily be integrated. [The multipliers are so chosen that the denominators become zero and the numerators become exact differentials.] Sometimes only one set of multipliers may fulfil the object.

Any combination of the methods given above will afford two independent relations between the variables which will constitute the general solution of (3). Each of these relations will contain an arbitrary constant.

9.5. Geometrical interpretation of the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

We know, from the geometry of three dimensions, that the direction cosines of the tangent to a curve are

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds},$$

that is, are in the ratio $dx : dy : dz$.

Hence geometrically these equations represent a system of curves in space, such that the direction cosines of the tangent to it at any point (x, y, z) are proportional to P, Q, R.

9.6. Illustrative Examples.

Ex. 1. *Solve* : $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$. [K. H. 2006]

Taking the first two members of the set of equations, we have

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

or, $$\frac{dx}{x} = \frac{dy}{y}.$$

Integrating both sides, we get

$$\log x = \log (C_1 y)$$

or, $$x = C_1 y. \quad \ldots (1)$$

Again, considering the last two members, we have

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

or, $$\frac{dy}{y^2} = \frac{dz}{C_1 y^2 z - 2C_1^2 y^2}, \text{ from (1)}$$

or, $$dy = \frac{dz}{C_1 z - 2C_1^2}$$

or, $$C_1\, dy = \frac{dz}{z - 2C_1}.$$

Integrating both sides, we get

$$C_1 y = \log (z - 2C_1) + C_2.$$

Therefore $$x = \log \left(z - \frac{2x}{y}\right) + C_2, \quad \text{from (1)}$$

or, $$x = \log (yz - 2x) - \log y + C_2. \quad \ldots (2)$$

(1) and (2) constitute the solution of the given equations.

CHAPTER X

TOTAL DIFFERENTIAL EQUATIONS

10.1. Introduction.

An equation of the form

$$P\,dx + Q\,dy + R\,dz = 0,$$

in which P, Q, R are functions of x, y and z, is called a *total differential equation* or *single differential equation* or *Pfaffian differential equation*.

If we are given a relation of the form

$$f(x, y, z) = C, \quad \ldots (1)$$

where C is an arbitrary constant, then we can write

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0. \quad \ldots (2)$$

If the quantities $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ have a common factor, then the relation (2) can be simplified by cancelling that factor throughout and (2) takes the form

$$P\,dx + Q\,dy + R\,dz = 0, \quad \ldots (3)$$

in which P, Q, R are functions of x, y and z.

Thus, if a relation (1) be given, then from it, we can find a relation of the type (3), which is a total differential equation.

We shall now try the converse, that is, if we are given a relation like (3), how to get the corresponding relation like (1) from it. Obviously, for arbitrary values of P, Q and R, it is not possible. We shall try to find the circumstances under which a total differential equation will lead to an integral of the type (1).

10.2. Condition of integrability of $P\,dx + Q\,dy + R\,dz = 0$.

Let us consider the equation

$$P\,dx + Q\,dy + R\,dz = 0, \quad \ldots (1)$$

in which P, Q, R are functions of x, y and z.

If it be integrable, let its integral be $f(x, y, z) =$ a constant, whose total differential df is equal to

$$P\,dx + Q\,dy + R\,dz$$

or this expression multiplied by a factor which may be a function of x, y and z.

But $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Hence we have $$\frac{\partial f}{\partial x} = \mu P, \quad \frac{\partial f}{\partial y} = \mu Q, \quad \frac{\partial f}{\partial z} = \mu R,$$

where the unknown quantity μ is same for all and is a function of x, y and z.

From the last two of these equations, we have

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial z\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)$$

or, $$\frac{\partial}{\partial z}(\mu Q) = \frac{\partial}{\partial y}(\mu R)$$

or, $$\mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} = \mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y}$$

or, $$\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z}.$$

Similarly, $$\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x}$$

and $$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y}.$$

Multiplying these three equations by P, Q and R respectively and adding them, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \quad \ldots (2)$$

This is the relation which must hold among P, Q, R in order that the equation (1) possesses an integral of the form

$$f(x, y, z) = \text{constant}.$$

Thus the condition (2) is *necessary*.

To prove that this condition is also *sufficient*, we assume that it is satisfied by the coefficients P, Q, R of the relation

$$P\,dx + Q\,dy + R\,dz = 0.$$

Then it can be easily verified that the relation (2) is satisfied by the coefficients of the relation

$$\mu P\,dx + \mu Q\,dy + \mu R\,dz = 0, \qquad \ldots (3)$$

where μ is a function of x, y and z.

If $(P\,dx + Q\,dy)$ be not an exact differential with respect to x and y, then an integrating factor μ may be found to make it exact, and (3) can then be taken as the equation to be considered. Hence, without any loss of generality, we can regard $(P\,dx + Q\,dy)$ as an exact differential so that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \qquad \ldots (4)$$

Let $\int (P\,dx + Q\,dy) = V$,

then $\dfrac{\partial V}{\partial x} = P$ and $\dfrac{\partial V}{\partial y} = Q$.

Hence $\dfrac{\partial P}{\partial z} = \dfrac{\partial^2 V}{\partial z\,\partial x}$ and $\dfrac{\partial Q}{\partial z} = \dfrac{\partial^2 V}{\partial z\,\partial y}$.

Substituting these values in the relation

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0,$$

it comes as

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\,\partial y} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z\,\partial x}\right) = 0, \quad \text{using (4)}$$

or, $$\frac{\partial V}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) + \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\left(R - \frac{\partial V}{\partial z}\right) = 0$$

or, $$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z} - R\right) \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0.$$

This equation shows that there is a relation between V and $\left(\frac{\partial V}{\partial z} - R\right)$, independent of x and y. Hence $\left(\frac{\partial V}{\partial z} - R\right)$ can be expressed as a function of V and z alone. Let

$$\frac{\partial V}{\partial z} - R = \phi(z, V), \text{ that is, } R = \frac{\partial V}{\partial z} - \phi(z, V).$$

Now $P\,dx + Q\,dy + R\,dz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz - \phi(z, V)\,dz$

$$= dV - \phi(z, V)\,dz.$$

Thus the equation $P\,dx + Q\,dy + R\,dz = 0$ becomes

$$dV - \phi(z, V)\,dz = 0,$$

which being an equation in two variables is integrable and its integral may be taken in the form

$$F(z, V) = 0.$$

Note. If the expression $(P\,dx + Q\,dy + R\,dz)$ be an exact differential, then the equation $P\,dx + Q\,dy + R\,dz = 0$ is called an *exact equation*. In that case, if the integral be $u(x, y, z) = C$, then obviously

$$P = \frac{\partial u}{\partial x},\ Q = \frac{\partial u}{\partial y} \text{ and } R = \frac{\partial u}{\partial z}.$$

Hence $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

10.3. Methods of solution of total differential equations.

When the condition of integrability is satisfied, we can use any one of the following methods to get the required integral of the given equation.

Method 1 : Solution by inspection.

It may be possible in many cases that, by re-arranging the terms, the equation becomes exact and the solution is found readily.

Method 2 : One variable regarded as constant.

If any two terms of the equation, say $P\,dx + Q\,dy = 0$ can easily be solved, then the third variable z is taken as constant, so that $dz = 0$.

Let the integral of $P\,dx + Q\,dy = 0$ be ... (1)

$$u = \text{constant} = \phi.$$

This constant in the integral is constant only with respect to x and y and can therefore be assumed to be a function of z. Taking differential on both sides of (1) and comparing the result with the equation

$$P\,dx + Q\,dy + R\,dz = 0,$$

we shall be able to determine the constant appearing in the integral (1) as a function of z. If the coefficients of $d\phi$ or dz involve functions of x and y, then it will be possible to eliminate them with the help of the integral (1).

Thus we shall get $\frac{d\phi}{dz}$ which is independent of x and y and which on integration will give ϕ.

Then (1) will be the complete solution of the equation.

Method 3 : Homogeneous equations.

In case P, Q, R are homogeneous functions of x, y and z, then one variable, say z, may be separated from the other two by the substitutions $x = zu$ and $y = zv$

so that $dx = z\,du + u\,dz$ and $dy = z\,dv + v\,dz$.

Putting these in the equation, the equation will be found to be reduced to the form in which either the coefficient of dz is zero or not zero. In either case the new equation may easily be integrated.

In some cases an integrating factor will be required to put the homogeneous equation in easily integrable form.

Let $P\,dx + Q\,dy + R\,dz = 0$ ... (1)

be an integrable equation in which P, Q, R are homogeneous functions of x, y, z of degree n. Let us substitute $x = uz$ and $y = vz$ so that $dx = u\,dz + z\,du$ and $dy = v\,dz + z\,dv$.

Also let $P = z^n f(u, v)$, $Q = z^n \phi(u, v)$ and $R = z^n \psi(u, v)$.

Putting these in (1), we get

$$z^n \{ f(u, v)(u\,dz + z\,du) + \phi(u, v)(v\,dz + z\,dv) + \psi(u, v)\,dz \} = 0.$$

Thus $z^n [z \{ f(u, v) du + \phi(u, v) dv \} + \{ u f(u, v) + v \phi(u, v) + \psi(u, v) \} dz] = 0.$

Let us divide this by

$$z^{n+1} \{ u f(u, v) + v \phi(u, v) + \psi(u, v) \},$$

if it be not equal to zero.

Thus we get

$$\frac{f(u, v) du + \phi(u, v) dv}{u f(u, v) + v \phi(u, v) + \psi(u, v)} + \frac{dz}{z} = 0. \quad \ldots (2)$$

Now, the equation (1) being integrable so will be the equation (2). Also, the variables of the first term of (2) are u and v while that of the second term is z. Thus the variable z is separated from other two variables of the equation. If we multiply (2) by any factor containing the variables u, v, z, then this separation will be destroyed. This suggests that (2) must be an exact equation in itself. But equation (2) was obtained by dividing the equation (1) by the factor

$$z^{n+1} \{ u f(u, v) + v \phi(u, v) + \psi(u, v) \}$$

besides the change of variables. This factor is $(Px + Qy + Rz)$.

Hence $\dfrac{1}{Px + Qy + Rz}$ is an integrating factor of the homogeneous equation (1), except for the case when $Px + Qy + Rz = 0$. In such cases the general method of solution is applied.

Method 4 : Auxiliary equations.

Let the given equation $P\,dx + Q\,dy + R\,dz = 0$ be integrable.

Then we have

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

Comparing these two, we get

$$\frac{dx}{\dfrac{\partial Q}{\partial z} - \dfrac{\partial R}{\partial y}} = \frac{dy}{\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}} = \frac{dz}{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}.$$

These are called *auxiliary equations* and can be solved by methods discussed earlier.

Let $u = a$ and $v = b$ be two integrals.

Now we wish to find A and B in such a way that the given equation can be written as $A\,du + B\,dv = 0$.

So we find $A\,du + B\,dv = 0$ and compare it with $P\,dx + Q\,dy + R\,dz = 0$.

Then using $u = a$ and $v = b$, we obtain the values of A and B in terms of u and v. With these values of A and B in $A\,du + B\,dv = 0$, we get the required integral on integration.

Note. This method is inapplicable, if the equation be exact, that is, if

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \quad \text{and} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

10.4. Geometrical interpretation of $P\,dx + Q\,dy + R\,dz = 0$.

This differential equation shows that two straight lines whose direction cosines are proportional to dx, dy, dz and P, Q, R are perpendicular to each other. Now the direction cosines of the tangent to a curve at a point (x, y, z) are proportional to dx, dy, dz. Hence the above equation expresses that the tangent to a curve at the point (x, y, z) is perpendicular to a straight line whose direction cosines are proportional to P, Q, R.

10.5. The locus of $P\,dx + Q\,dy + R\,dz = 0$.

We have seen that, for a point moving along a curve subject to the condition

$$P\,dx + Q\,dy + R\,dz = 0, \quad \ldots (1)$$

the direction in which it moves is at right angles to the straight line whose direction cosines are proportional to P, Q, R.

We further know that a straight line whose direction cosines are proportional to dx, dy, dz is parallel to a straight line whose direction cosines are proportional to P, Q, R under the condition

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \quad \ldots (2)$$

Thus we see that the curves traced out by points which move subject to the condition (1) are orthogonal to the curves traced out by points which move subject to the condition (2).

If (1) be integrable, then a family of surfaces can be found that are normal to the curves given by (2) at the points where these curves cut the surface.

If, on the other hand, (1) be non-integrable, then no family of surfaces can be found which is orthogonal to all lines that form the locus of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

10.6. Equations having more than three variables.

Let us consider an equation of the form

$$P\,dx + Q\,dy + R\,dz + S\,dt = 0, \qquad \ldots (1)$$

where P, Q, R, S are functions of x, y, z and t.

It is integrable when any one of the four variables is made constant. Taking x to be a constant so that $dx = 0$, the equation (1) becomes

$$Q\,dy + R\,dz + S\,dt = 0. \qquad \ldots (2)$$

If the equation (1) be integrable, then the equation (2) will also be integrable. The condition of integrability of (2) is

$$Q\left(\frac{\partial R}{\partial t} - \frac{\partial S}{\partial z}\right) + R\left(\frac{\partial S}{\partial y} - \frac{\partial Q}{\partial t}\right) + S\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = 0.$$

Similarly, taking y, z, t successively one by one as constants, the conditions of integrability are

$$P\left(\frac{\partial R}{\partial t} - \frac{\partial S}{\partial z}\right) + R\left(\frac{\partial S}{\partial x} - \frac{\partial P}{\partial t}\right) + S\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) = 0,$$

$$P\left(\frac{\partial Q}{\partial t} - \frac{\partial S}{\partial y}\right) + Q\left(\frac{\partial S}{\partial x} - \frac{\partial P}{\partial t}\right) + S\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

and $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

respectively.

It is observed that the fourth condition can be derived from the first three. Hence it is not independent. In fact, any of these four conditions can be obtained from the remaining three.

Thus, for an equation (1) involving four variables, the conditions of integrability must hold for the coefficients of all the terms taken by threes.

When the conditions of integrability are satisfied, the integral is found out as in the case of three variables treating here two of the variables as constants. The constant of integration here is taken to be a function of two variables which were regarded as constants earlier. Then differential of the integral relation is formed. Comparing it with the given equation, a relation free of two variables and their differentials are obtained. Then the solution is obtained as in the general case.

Note. Proceeding similarly, the conditions of integrability can be derived for an equation having any finite number of independent variables.

In case of n independent variables, the number of independent conditions of integrability is $\frac{1}{2}(n-1)(n-2)$.

10.7. Non- integrable single differential equation.

Let us consider an equation

$$P\,dx+Q\,dy+R\,dz=0, \quad \ldots \quad (1)$$

for which the condition of integrability is not satisfied.

Thus there exists no single relation among the variables x, y, z to satisfy it.

Let us assume an integral relation

$$f(x, y, z)=0, \quad \ldots \quad (2)$$

which on differentiation gives

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0. \quad \ldots \quad (3)$$

When the form $f(x, y, z)$ is specified, from (2) and (3) we can eliminate one of the variables and its differential from (1) and it becomes of the form $M\,dx+N\,dy=0$, where M and N are functions of x and y, if z and dz be eliminated. Solving this, we get an equation involving an arbitrary constant. This equation, together with (2), will constitute the solution of (1).

Note. Here every possible solution can be obtained from different forms of (2).

10.8. Illustrative Examples.

Ex. 1. *Solve :* $(y+z)\,dx + dy + dz = 0$.

Here $P = y+z,\ Q = R = 1$.

Therefore $P\left(\dfrac{\partial Q}{\partial z} - \dfrac{\partial R}{\partial y}\right) + Q\left(\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}\right) + R\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right)$

$$= (y+z)\cdot 0 + 1\cdot(-1) + 1\cdot(1) = 0.$$

Thus the condition of integrability is satisfied.

The given equation can be written as

$$dx + \frac{d(y+z)}{y+z} = 0.$$

Integrating, we get

$$x + \log(y+z) = C.$$

This is the complete integral of the given equation.

Ex. 2. *Solve :* $(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0$. [*B. H. 1998*]

Here $P = y^2 + yz$, $Q = z^2 + zx$ and $R = y^2 - xy$.

The condition of integrability is satisfied.

Let z be a constant, so that $dz = 0$. Hence the given equation reduces to

$$(y^2 + yz)\,dx + (z^2 + zx)\,dy = 0$$

or, $$y(y+z)\,dx + z(z+x)\,dy = 0$$

or, $$\frac{dx}{z+x} + \frac{z}{y(y+z)}\,dy = 0$$

or, $$\frac{dx}{z+x} + \left(\frac{1}{y} - \frac{1}{y+z}\right)dy = 0$$

Integrating, we get

$$\log(z+x) + \log y - \log(y+z) = \text{constant}$$

or, $$\frac{y(z+x)}{y+z} = \text{constant, independent of } x \text{ and } y \qquad \ldots (1)$$

$$= \phi, \text{ say,}$$

where ϕ is a function of z.

Taking differential on both sides, we get

$$\frac{(y+z)\{y(dz+dx)+(z+x)dy\}-y(z+x)(dy+dz)}{(y+z)^2}=d\phi$$

or, $$\frac{(y^2+yz)\,dx+(z^2+zx)\,dy+(y^2-xy)\,dz}{(y+z)^2}=d\phi.$$

Thus $d\phi=0$, using the given equation

or, $\phi=C$, C being a constant

or, $$\frac{y(z+x)}{y+z}=C$$

or, $$y(z+x)=C(y+z).$$

This is the complete solution of the given equation.

Ex. 3. *Solve :* $z^2\,dx+(z^2-2yz)\,dy+(2y^2-yz-zx)\,dz=0.$

Here the condition of integrability is satisfied.

Let z be a constant, so that $dz=0$, then the given equation reduces to

$$z^2\,dx+(z^2-2yz)\,dy=0.$$

Integrating, we get

$$z^2x+z^2y-y^2z=\text{constant}=\phi\ (\text{say}),$$

where ϕ is a function of z.

Taking differential on both sides, we get

$$z^2\,dx+(z^2-2yz)\,dy+(2zx+2zy-y^2)\,dz=d\phi=\frac{d\phi}{dz}\,dz.$$

Comparing this with the given equation, we have

$$2zx+2zy-y^2-\frac{d\phi}{dz}=2y^2-yz-zx$$

or, $$3(zx+yz-y^2)\,dz=d\phi$$

or, $$\frac{3}{z}(z^2x+z^2y-y^2z)\,dz=d\phi.$$

Hence $$\frac{3\phi}{z}\,dz=d\phi$$

or, $$\frac{d\phi}{\phi}=\frac{3\,dz}{z}.$$

Integrating both sides, we get

$$\log\phi=3\log z+\log C$$

or, $$\phi=Cz^3.$$

Thus $$z^2x + z^2y - y^2z = \phi = Cz^3$$

or, $$zx + zy - y^2 = Cz^2.$$

This is the complete integral of the given equation.

Ex. 4. *Solve* : $(2xz - yz)\,dx + (2yz - zx)\,dy - (x^2 - xy + y^2)\,dz = 0.$

Condition of integrability is satisfied ; for, here

$$P = 2xz - yz,\ Q = 2yz - zx \text{ and } R = -x^2 + xy - y^2.$$

Moreover P, Q, R are homogeneous functions of x, y, z.

Put $x = uz$ and $y = vz$ so that $dx = u\,dz + z\,du$ and $dy = v\,dz + z\,dv$.

Substituting these in the given equation, we get the equation as

$$(2z^2u - z^2v)(u\,dz + z\,du) + (2z^2v - z^2u)(v\,dz + z\,dv) - (z^2u^2 - z^2uv + z^2v^2)\,dz = 0$$

or, $$(2u - v)(u\,dz + z\,du) + (2v - u)(v\,dz + z\,dv) - (u^2 - uv + v^2)\,dz = 0$$

or, $$z(2u - v)\,du + z(2v - u)\,dv + (u^2 + v^2 - uv)\,dz = 0$$

or, $$\frac{2u - v}{u^2 + v^2 - uv}\,du + \frac{2v - u}{u^2 + v^2 - uv}\,dv + \frac{dz}{z} = 0$$

or, $$\frac{(2u - v)\,du + (2v - u)\,dv}{u^2 + v^2 - uv} + \frac{dz}{z} = 0.$$

Integrating, we get

$$\log(u^2 + v^2 - uv) + \log z = \log C$$

or, $$z(u^2 + v^2 - uv) = C$$

or, $$z\left(\frac{x^2}{z^2} + \frac{y^2}{z^2} - \frac{xy}{z^2}\right) = C$$

or, $$x^2 + y^2 - xy = Cz.$$

This is the required solution.