ADVANCED ANALYTICAL GEOMETRY

TWO DIMENSIONS

1 TRANSFORMATION OF CO-ORDINATES

1.1. Introduction.

Equations of curves or straight lines have reference to certain set of axes. In fact, we shall have different equations for the same curve or straight line whan referred to different set of co-ordinate axes. This may happen when the origin is shifted to a point keeping the directions of the axes the same or when the axes are rotated through the same angle keeping the origin unaltered. The former is called *translation* and the latter is called *rotation*. Change of co-ordinates may also be effected by a combination of the two, in either order and is called a *rigid motion*. These transformations are also known as *orthogonal transformations*.

A general orthogonal transformation is one in which a new origin is taken and two perpendicular lines through the new origin are taken as the axes, whose equations referred to the old axes are known.

1.2. Change of origin without changing the directions of the axes (*Translation*).

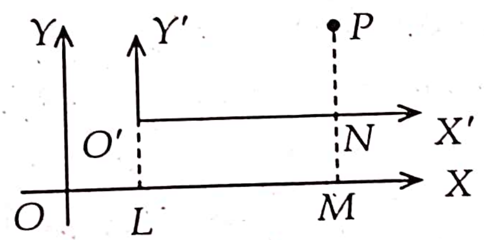

Fig. 1.

Let OX, OY be the set of rectangular axes referred to which the co-ordinates of an arbitrary point P are (x, y). Let O', the new origin, be at (h, k) and $O'X'$, $O'Y'$ be the new set of axes parallel to the original axes. Let the co-ordinates of P referred to the new set of axes be (x', y'). Then

$$x = OM = OL + LM = OL + O'N = h + x',$$

$$y = PM = MN + NP = O'L + PN = k + y'.$$

Hence $x = x' + h$ and $y = y' + k$

are the transformation formulæ for the translation of axes.

Thus the equation $f(x, y) = 0$ with reference to the old set of axes becomes $f(x' + h, y' + k) = 0$ with reference to the new set of axes. Removing the primes to put it in general form, the new equation becomes $f(x + h, y + k) = 0$.

Note. The above formulæ may also be written as

$$x' = x - h \text{ and } y' = y - k.$$

1.3. Transformation from one pair of rectangular axes to another with the same origin (*Rotation*).

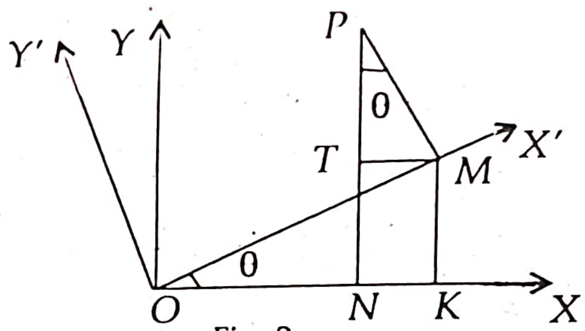

Fig. 2

Let the axes OX and OY be turned about O through an angle θ to the position OX' and OY'. Let P be any point (x, y) referred to the system OX, OY and (x', y') referred to the new set of axes OX', OY'.

PN and PM are drawn perpendiculars to OX and OX' respectively. Draw MK and MT perpendiculars to OX and PN respectively.

We have $OM = x'$ and $PM = y'$.

Then $x = ON = OK - NK = OK - MT = x' \cos\theta - y' \sin\theta$,

since $\angle TPM = 90° - \angle TMP = \angle TMO = \theta$.

Again $y = PN = TN + PT = MK + PT = x' \sin\theta + y' \cos\theta$.

Hence

$$x = x' \cos\theta - y' \sin\theta,\ y = x' \sin\theta + y' \cos\theta \quad \ldots \quad (1)$$

are the transformation formulæ for the rotation of axes.

Note. To find x', y' in terms of x, y, we are to write $(-\theta)$ for θ and to interchange x and x', y and y' { or solve equations (1) }.

Thus $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$. $\quad \ldots (2)$

The transformations of co-ordinations given by (1) and (2) are expressed by the scheme

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

1.4. Translation followed by a rotation.

When the origin is shifted to the point (h, k), the co-ordinates of any point $P(x, y)$ become $(x + h, y + k)$. Then the axes are turned through an angle θ. The co-ordinates of P referred to the new set of axes are obtained by substituting $(x\cos\theta - y\sin\theta)$ for x and $(x\sin\theta + y\cos\theta)$ for y.

Hence the co-ordinates of P due to a rigid motion become

$$(h + x\cos\theta - y\sin\theta,\ k + x\sin\theta + y\cos\theta).$$

As is obvious, the result will be the same, if rotation be followed by translation.

Cor. The general formulæ for transformation of axes may therefore be written as

$$x = px' - qy' + h,\ y = qx' + py' + k$$

where $p^2 + q^2 = 1$.

1.5. General orthogonal transformation.

Consider the system of rectangular axes, OX and OY (that is, OXY) O being the origin. Let O' be the origin of the new set of rectangular axes $O'X'Y'$, such that the equations of $O'X'$ and $O'Y'$ referred to the set OXY are respectively $ax + by + c = 0$ and $bx - ay + c' = 0$.

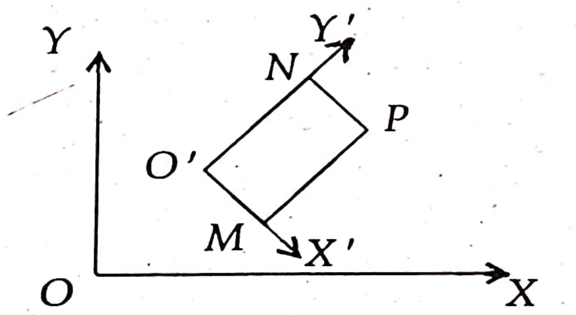

Fig. 3

Let P be a point coplanar with the two systems of axes whose co-ordinates are (x, y) referred to the set OXY and (x', y') referred to the set $O'X'Y'$. Then

$$x' = O'M = PN = \frac{bx - ay + c'}{\sqrt{a^2 + b^2}} \quad \text{and} \quad y' = PM = \frac{ax + by + c}{\sqrt{a^2 + b^2}}.$$

Solving these two, we get the general orthogonal transformation formulæ as

$$x = \frac{bx' + ay'}{\sqrt{a^2 + b^2}} - \frac{ac + bc'}{a^2 + b^2} \text{ and } y = \frac{-ax' + by'}{\sqrt{a^2 + b^2}} - \frac{bc - ac'}{a^2 + b^2}.$$

Note. Thus we see that, in the general orthogonal transformation, we substitute for x and y expressions in x' and y' of the first degree. Hence, by this substitution, the degree of the equation is not altered, since the transformation is linear.

1.6. Invariants in orthogonal transformation.

Relations connecting the coefficients of an expression or some other quantity which remain unchanged under an orthogonal transformation are called *invariants* under that orthogonal transformation.

If, by the orthogonal transformation without change of origin, the expression $(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$ *be changed into* $(a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'Y + c')$, *then*

(*i*) $a' + b' = a + b$; (*ii*) $a'b' - h'^2 = ab - h^2$;

(*iii*) $g'^2 + f'^2 = g^2 + f^2$;

(*iv*) $b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2$
$= bc + ca + ab - f^2 - g^2 - h^2$.

By applying the formulæ for orthogonal transformation, *viz.*

$$x = X\cos\theta - Y\sin\theta,\ y = X\sin\theta + Y\cos\theta,$$

the given expression changes to

$$a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ b(X\sin\theta + Y\cos\theta)^2 + 2g(X\cos\theta - Y\sin\theta)$$
$$+ 2f(X\sin\theta + Y\cos\theta) + c$$
$$= X^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$+ 2XY\{h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta\}$$
$$+ Y^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)$$
$$+ 2X(g\cos\theta + f\sin\theta) + 2Y(f\cos\theta - g\sin\theta) + c$$

so that

$$a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta,$$
$$h' = h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta,$$
$$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta,$$
$$g' = g\cos\theta + f\sin\theta,\ f' = f\cos\theta - g\sin\theta,\ c' = c.$$

Hence we have $a' + b' = a + b,$ ... (*i*)

since $\sin^2\theta + \cos^2\theta = 1.$

Again $2a' = a(1 + \cos 2\theta) + 2h\sin 2\theta + b(1 - \cos 2\theta)$

$$= (a + b) + \{2h\sin 2\theta + (a - b)\cos 2\theta\}.$$

Similarly, $2b' = (a + b) - \{2h\sin 2\theta + (a - b)\cos 2\theta\}.$

Therefore $4a'b' = (a + b)^2 - \{2h\sin 2\theta + (a - b)\cos 2\theta\}^2.$

Also $4h'^2 = \{2h\cos 2\theta - (a - b)\sin 2\theta\}^2.$

Therefore $4(a'b' - h'^2) = (a + b)^2 - \{4h^2 + (a - b)^2\}$

$$= 4(ab - h^2).$$

Hence $a'b' - h'^2 = ab - h^2.$ ... (*ii*)

Again $g'^2 + f'^2 = (g\cos\theta + f\sin\theta)^2 + (f\cos\theta - g\sin\theta)^2$

$$= g^2 + f^2.$$ ... (*iii*)

Lastly $b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2$

$$= c'(a' + b') + (a'b' - h'^2) - (f'^2 + g'^2)$$

$$= c(a + b) + (ab - h^2) - (f^2 + g^2)$$

$$= bc + ca + ab - f^2 - g^2 - h^2.$$ ... (*iv*)

1.7. Illustrative Examples.

Ex. 1. *Find the transformed equation of the straight line* $\frac{x}{a} + \frac{y}{b} = 2$ *when the origin is transferred to the point (a, b).*

The transformation formulæ are $x = x' + a$, $y = y' + b$ so that the transformed equation is

$$\frac{1}{a}(x' + a) + \frac{1}{b}(y' + b) = 2, \text{ that is, } \frac{x'}{a} + \frac{y'}{b} = 0.$$

Ex. 2. *Transform to parallel axes through the point* (2, − 3) *the equation*

$$2x^2 + 4xy + 3y^2 - 2x - 4y + 7 = 0.$$

The transformation is effected by $x = x' + 2$, $y = y' - 3$;

so that the transformed equation is

$$2(x' + 2)^2 + 4(x' + 2)(y' - 3) + 3(y' - 3)^2 - 2(x' + 2) - 4(y' - 3) + 7 = 0$$

or, $2x'^2 + 4x'y' + 3y'^2 - 6x' - 14y' + 26 = 0$

Ex. 3. *Transform to axes inclined at 30° to the original axes the equation*

$$x^2 + 2\sqrt{3}\,xy - y^2 - 2 = 0.$$

The transformation formulæ are

$$x = x' \cos 30° - y' \sin 30° = \tfrac{1}{2}(x'\sqrt{3} - y'),$$

$$y = x' \sin 30° + y' \cos 30° = \tfrac{1}{2}(x' + y'\sqrt{3}).$$

The transformed equation is

$$(x'\sqrt{3} - y')^2 + 2\sqrt{3}\,(x'\sqrt{3} - y')(x' + y'\sqrt{3}) - (x' + y'\sqrt{3})^2 = 8$$

or, $x'^2 - y'^2 = 1.$

Ex. 4. *To what point the origin is to be moved so that one can get rid of the first degree terms from the equation*

$$x^2 + xy + 2y^2 - 7x - 5y + 12 = 0\ ?$$

Let the point to which the origin is to be shifted be (α, β).

Substituting $x = x' + \alpha$, $y = y' + \beta$, the equation becomes

$$(x' + \alpha)^2 + (x' + \alpha)(y' + \beta) + 2(y' + \beta)^2 - 7(x' + \alpha) - 5(y' + \beta) + 12 = 0.$$

The coefficients of x' and y' in the transformed eqution are

$$(2\alpha + \beta - 7) \text{ and } (\alpha + 4\beta - 5),$$

which will be separately zero, if the first degree terms are to be removed.

Thus $\quad 2\alpha + \beta - 7 = 0$ and $\alpha + 4\beta - 5 = 0.$

Solving, we get $\alpha = 3\frac{2}{7}$, $\beta = \frac{3}{7}$.

Hence the origin must be shifted to the point $(3\frac{2}{7}, \frac{3}{7})$.

Note. To remove the linear terms, the origin is to be shifted.

Ex. 5. *Find the angle through which the axes are to be rotated so that the equation $x\sqrt{3} + y + 6 = 0$ may be reduced to the form $x = c$. Also determine the value of c.*

Let the axes be rotated through an angle θ, so that

$$x = x'\cos\theta - y'\sin\theta \text{ and } y = x'\sin\theta + y'\cos\theta.$$

The reduced equation becomes, after removing the primes,

$$\sqrt{3}\,(x\cos\theta - y\sin\theta) + (x\sin\theta + y\cos\theta) + 6 = 0$$

or, $\quad x(\sqrt{3}\cos\theta + \sin\theta) + y(\cos\theta - \sqrt{3}\sin\theta) + 6 = 0.$

Since this is to be of the form $x = c$, therefore

$$\cos\theta - \sqrt{3}\sin\theta = 0, \text{ whence } \theta = \tfrac{1}{6}\pi.$$

Thus the transformed equation is

$$x\left(\frac{3}{2} + \frac{1}{2}\right) = -6$$

or, $\quad x = -3.$

Hence $\quad c = -3.$

Ex. 6. *Find the angle by which the axes should be rotated so that the equation $ax^2 + 2hxy + by^2 = 0$ becomes another equation in which the term xy is absent.*

In particular, find the angle through which the axes are to be rotated so that the equation

$$17x^2 + 18xy - 7y^2 = 1$$

may be reduced to the form $Ax^2 + By^2 = 1$, $A > 0$;

find also A and B.

Let the axes be turned through an angle θ.

Then, after removing the primes from (x', y'), the general form of the equation becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 = 0.$$

The coefficient of xy will be zero, if

$$(b - a)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) = 0,$$

that is, $\quad -\frac{1}{2}(a - b)\sin 2\theta + h\cos 2\theta = 0,$

that is, $\quad \tan 2\theta = \dfrac{2h}{a - b}$, giving $\theta = \frac{1}{2}\tan^{-1}\dfrac{2h}{a - b}$.

In the given case, $a = 17$, $h = 9$, $b = -7$.

Therefore $\theta = \frac{1}{2}\tan^{-1}\dfrac{18}{17 + 7} = \frac{1}{2}\tan^{-1}\dfrac{3}{4}$.

By invariants, we get

$A + B = 17 - 7 = 10$ and

$AB = 17(-7) - 9^2 = -200$

Therefore $(A - B)^2 = (A + B)^2 - 4AB = 900$

or, $\quad A - B = \pm 30.$

Hence $A = 20$, $B = -10$ $(\because A > 0)$,

and the transformed equation is $20x^2 - 10y^2 = 1$.

Note. To remove the xy-term, the axes are to be rotated.

More generally, to remove any second degree term, *e.g.*, x^2, y^2, xy, the axes are to be rotated.

Examples I

1. (*a*) The origin is shifted to the point $(2, -3)$ without changing the directions of the axes. Find the co-ordinates of the point referred to the new set of axes, if its co-ordinates with respect to the old set of axes be $(3, 1)$. Also find the co-ordinates of another point in the old system, if its co-ordinates in the new system become $(-4, 8)$.

(*b*) The axes are rotated through an angle of 30° with the same origin. Find the new co-ordinates of the point, whose old co-ordinates were $(2, 4)$. Also find the old co-ordinates of the point, whose new co-ordinates will be $(-4, 2)$.

2. The origin is shifted to the point $(3, -1)$ and the co-ordinate axes are then rotated through an angle $\tan^{-1}\frac{3}{4}$. Find the co-ordinates of the point $(5, -2)$ in the new co-ordinate system. [*B. H.* 1997]

If the co-ordinates of a point be $(5, 10)$ in the new system, find its old co-ordinates. [*N. B. H.* 2002]

3. Find the equation to the curve

$$9x^2 + 4y^2 + 18x - 16y = 11$$

referred to parallel axes through the point $(-1, 2)$.

4. (*a*) Transform the equations

$$(i)\ x^2 - y^2 = a^2\ ; \quad (ii)\ x^2 + 2axy + y^2 = a^2$$

to axes inclined at 45° to the original axes.

(*b*) What does the equation $11x^2 + 16xy - y^2 = 0$ become on turning the axes through an angle $\tan^{-1}\frac{1}{2}$?

(*c*) Find the equation of the straight line $x \cos\alpha + y \sin\alpha = p$ when the axes are turned through an angle α without any change of origin.

5. (*a*) What does the equation

$$x^2 - 3xy + 3y^2 + 7x - 18y + 32 = 0$$

become when the origin is moved to the point $(4, 5)$ and the axes are turned through an angle of 45° ?

(*b*) Find the result when the origin is shifted to the point $(1, 2)$ and the axes are turned through an angle of 45° for the equation $x^2 - 2xy - 3y^2 + 2x + 14y - 16 = 0$.

6. Reduce the equations $x - y + 3 = 0$ and $2x - y + 1 = 0$ to the form $ax' + by' = 0$ by proper choice of origin and find this new origin.

7. (*a*) To what points must the origin be moved in order to remove the terms of the first degree in the equations

(*i*) $2x^2 - 3y^2 - 4x - 12y = 0$;

(*ii*) $2x^2 - 3xy + 4y^2 + 10x - 19y + 23 = 0$.

(*b*) Choose a new origin (h, k) without changing the directions of the axes such that the equation $5x^2 - 2y^2 - 30x + 8y = 0$ may be reduced to the form $Ax'^2 + By'^2 = 1$.

(*c*) Find the translation which transforms the equation $x^2 + y^2 - 2x + 14y + 20 = 0$ into $x'^2 + y'^2 - 30 = 0$.

(*d*) Find the equations of the rigid motion that transforms $x^2 + 2x + y^2 - 10y + 25 = 0$ to a circle about the new origin as centre.

8. By transforming to parallel axes through a properly chosen point (h, k), prove that the following equation can be reduced to one containing only terms of the second degree :

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0.$$

Find the values of h and k.

9. (*a*) Prove that the linear part of the equation

$$4x^2 - 12xy + 9y^2 + 4x + 6y + 1 = 0$$

cannot be made to disappear by only change of parallel axes.

(*b*) By transforming to parallel axes through a properly chosen point (h, k), prove that the equation

$$2x^2 + 4xy + 5y^2 - 4x - 22y + 29 = 0$$

can be reduced to one containing only terms of the second degree. Also find the chosen point.

10. Without altering the directions of axes, change the origin to a suitable position so that the pair of points (3, 4) and (6, 8) may be represented by the co-ordinates of the form $(-\alpha, -\beta)$ and (α, β).

11. (*a*) Find the angle through which the axes must be turned so that the equation $lx - my + n = 0, (m \neq 0)$ may be reduced to the form $ay + b = 0$. [C. H. 2008]

(*b*) Find the angle of rotation about the origin which will transform the equation $x^2 - y^2 = 4$ into $x'y' + 2 = 0$.

12. Through what angle must the axes be turned to remove the term

(i) xy from $7x^2 + 4xy + 3y^2$;

(ii) xy from $x^2 + 2\sqrt{3}xy - y^2 = 4$;

(iii) xy from $9x^2 - 2\sqrt{3}xy + 7y^2 = 0$;

(iv) x^2 from $x^2 - 4xy + 3y^2 = 0$.

13. (a) The equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ is transformed to $4x^2 + 2y^2 = 1$ when referred to the rectangular axes through the point (2, 3). Find the inclination of the latter axes to the former.

(b) Find the value of a for which the transformation $x = \frac{1}{3}x' + 6y' + a,\ y = \frac{1}{3}x' + 3y' + a$ will transform the equation $x^2 - 2xy + y^2 - 3x + 6y - 12 = 0$ into $y'^2 = -\frac{1}{9}x'$. [B. H. 2006]

14. Show that there is only one point whose co-ordinates remain the same by changing the axes of co-ordinates by the transformation

$$x' = \frac{4}{5}x + \frac{3}{5}y - 2, \quad y' = -\frac{3}{5}x + \frac{4}{5}y + 2$$

and that point is (2, 4).

15. The points $A(0, 1)$, $B(-1, 0)$ and $C(1, 0)$ are transformed into the points A', B', C' respectively by the transformation

$$x' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y + 1, \quad y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y + 2.$$

Find the area of the triangle $A'B'C'$. [B. H. 1982]

16. $P(9, -1)$ and $Q(-3, 4)$ are two points. If the origin be shifted to the point P and the co-ordinate axes be rotated so that the positive direction of the new x-axis agrees with the direction of the segment $\overline{PQ}$, then find the co-ordinate transformation formulæ. [B. H. 1989]

17. Transform the equation

$$2x^2 - xy + y^2 + 2x - 3y + 5 = 0$$

to new axes of x and y given by the straight lines

$4x + 3y + 1 = 0$ and $3x - 4y + 2 = 0$ respectively.

18. If the perpendicular straight lines $ax + by + c = 0$ and $bx - ay + c' = 0$ be taken as the axes of x and y respectively, then show that the equation

$$(ax + by + c)^2 - 2(bx - ay + c')^2 = 1$$

will be transformed into $y'^2 - 2x'^2 = \dfrac{1}{a^2 + b^2}$.

19. (*a*) When the axes are turned through an angle, the expression $(ax + by)$ becomes $(a'x' + b'y')$ referred to new axes; show that

$$a^2 + b^2 = a'^2 + b'^2.$$

(*b*) If, by a rotation of rectangular axes about the origin, $(ax + by)$ and $(cx + dy)$ be changed to $(a'x' + b'y')$ and $(c'x' + d'y')$ respectively, then show that $ad - bc = a'd' - b'c'$.

[*B. H. 1991 ; N. B. H. 1992*]

20. If, by a rotation of rectangular axes about the origin, the expression $(ax^2 + 2hxy + by^2)$ changes to $(a'x'^2 + 2h'x'y' + b'y'^2)$, then prove that

$$a + b = a' + b' \quad \text{and} \quad ab - h^2 = a'b' - h'^2.$$ [*B.H. 1994*]

21. If, by a rotation of rectangular axes about the origin,

$(ax^2 + 2hxy + by^2)$ and $(cx^2 + 2gxy + dy^2)$ be changed to

$(a'x'^2 + 2h'x'y' + b'y'^2)$ and $(c'x'^2 + 2g'x'y' + d'y'^2)$ respectively, then show that

$$ad + bc - 2hg = a'd' + b'c' - 2h'g' \text{ and } ac + bd + 2hg = a'c' + b'd' + 2h'g'.$$

22. (*a*) Show that the distance between two fixed points and the area of the triangle formed by three fixed points are unaltered by (*i*) a translation of origin, (*ii*) a rotation of axes.

(*b*) Show that the discriminant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

of the quadratic expression $(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$ is invariant under (*i*) translation and (*ii*) rotation of axes. [*N. B. H. 2005*]

23. (*a*) Show that the radius of a circle remains unchanged due to a rigid body motion.

(*b*) Show that there is one point whose co-ordinates do not alter due to a rigid motion.

24. If $A_i\,(x_i\,,\,y_i\,),\ i=1,2,3$ be three points in a plane, then show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

remains invariant under a rigid motion. [B.H. 1995]

25. Prove that the transformation of rectangular axes which converts

$$\left(\frac{X^2}{p}+\frac{Y^2}{q}\right) \text{ into } (ax^2+2hxy+by^2) \text{ will convert}$$

$$\left(\frac{X^2}{p-\lambda}+\frac{Y^2}{q-\lambda}\right) \text{ into } \frac{ax^2+2hxy+by^2-\lambda(ab-h^2)(x^2+y^2)}{1-(a+b)\,\lambda+(ab-h^2)\,\lambda^2}.$$

Answers

1. (*a*) (1, 4); (− 2, 5).
(*b*) ($\sqrt{3}+2,\ 2\sqrt{3}-1$); ($-2\sqrt{3}-1,\ \sqrt{3}-2$). **2.** (1, − 2); (1, 10).

3. $9x'^2+4y'^2=36$.

4. (*a*) (*i*) $2x'y'+a^2=0$. (*ii*) $x'^2(1+a)+y'^2(1-a)=a^2$.,
(*b*) $3x'^2-y'^2=0$. (*c*) $x'=p$.

5. (*a*) $x'^2+4x'y'+7y'^2+2=0$.
(*b*) $2x'^2+4x'y'+1=0$.

6. $x'-y'=0,\ 2x'-y'=0$; (2 , 5).

7. (*a*) (*i*) (1, − 2) . (*ii*) (− 1, 2) .
(*b*) (3, 2) . (*c*) $x=x'+1,\ y=y'-7$.
(*d*) $x=x'\cos\theta-y'\sin\theta-1,\ \ y=x'\sin\theta+y'\cos\theta+5$.

8. $h=-\frac{3}{2},\ k=-\frac{5}{2}$. **9.** (*b*) (− 2, 3). **10.** $(\frac{9}{2},\ 6)$.

11. (*a*) $\tan^{-1}(l/m)$. (*b*) 45° .

12. (*i*) $\frac{1}{8}\pi$. (*ii*) 30° . (*iii*) $\frac{1}{3}\pi$. (*iv*) 45° .

13. (*a*) $\frac{1}{4}\pi$. (*b*) $a=4$. **15.** 1 sq. unit.

16. $x=\frac{1}{13}(-12x'-5y'+117),\ y=\frac{1}{13}(5x'-12y'-13)$.

17. $46x^2+29y^2+31xy+47x-29y+101=0$.

10 General Equation of Second Degree

10.1. Metric classification of conics.

Classification of conics by transformation of equations with the aid of translation and rotation of the axes is known as *metric classification of conics*.

A curve, which is represented by an equation of the second degree in cartesian co-ordinate system, is called a *curve of the second order*. An equation of a second order curve is said to be in its *canonical form* or *normal form*, if, by the transformation of co-ordinates, (*i*) the term in xy be removed from the equation, (*ii*) the number of terms in x and y be reduced to a minimum or be completely removed, (*iii*) the constant term be removed, if possible.

Consider the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \ldots \quad (1)$$

We shall show that this equation will represent a conic. Let us first transform the equation by rotation of axes such that we can get rid of the term xy. For this, we rotate the co-ordinate axes through an angle θ, that is, we put

$$x = x'\cos\theta - y'\sin\theta$$

and $$y = x'\sin\theta + y'\cos\theta,$$

the new co-ordinates being (x', y').

Then applying this transformation, the equation (1) becomes

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$
$$+ b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta)$$
$$+ 2f(x'\sin\theta + y'\cos\theta) + c = 0$$

or, $$x'^2(a\cos^2\theta + h\sin 2\theta + b\sin^2\theta)$$
$$+ x'y'\{2h\cos 2\theta - (a - b)\sin 2\theta\}$$
$$+ y'^2(a\sin^2\theta - h\sin 2\theta + b\cos^2\theta)$$
$$+ 2x'(g\cos\theta + f\sin\theta)$$
$$+ 2y'(f\cos\theta - g\sin\theta) + c = 0. \quad \ldots \quad (2)$$

Now let us choose θ such that the coefficient of the term containing $x'y'$ in (2) vanishes, that is,

$$2h\cos 2\theta - (a-b)\sin 2\theta = 0$$

or,
$$\tan 2\theta = \frac{2h}{a-b}. \qquad \ldots (3)$$

Whatever may be the values of a, b and h, there is always a value of θ satisfying (3). Choosing such a value for θ, (1) reduces to the form

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c' = 0. \qquad \ldots (4)$$

The following cases may arise :

Case I. *Let* $a' \neq 0$ *and* $b' \neq 0$.

Then the equation (4) can be put as

$$a'\left(x' + \frac{g'}{a'}\right)^2 + b'\left(y' + \frac{f'}{b'}\right)^2 = \frac{g'^2}{a'} + \frac{f'^2}{b'} - c' = k \text{ (say)}.$$

Shifting the origin to the point $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ without changing the directions of the axes, this equation becomes

$$a'X^2 + b'Y^2 = k, \qquad \ldots (5)$$

where (X, Y) are the new co-ordinates.

(*a*) If $k = 0$, then (5) represents *two straight lines,* which are real or imaginary according as a and b' are of opposite or of same sign.

A pair of imaginary straight lines is also called a *point ellipse* ; for, the equation is satisfied by the only point (0, 0).

(*b*) If $k \neq 0$, then (5) can be put as

$$\frac{X^2}{k/a'} + \frac{Y^2}{k/b'} = 1.$$

(*i*) If both $\frac{k}{a'}$ and $\frac{k}{b'}$ be positive, then (5) represents *an ellipse,* in its canonical form.

In addition, if $a' = b'$, then (5) represents a circle.

(*ii*) If one of $\frac{k}{a'}$ and $\frac{k}{b'}$ be positive and the other be negative, then (5) represents a *hyperbola,* in its canonical form.

In addition, if $a' + b' = 0$, then (5) represents *a rectangular hyperbola.*

(*iii*) If both of $\frac{k}{a'}$ and $\frac{k}{b'}$ be negative and k be positive, then (5) represents an *imaginary ellipse* ; for, there are no real values of x and y satisfying that equation.

Case II. *Let one of* a' *and* b', *say* a', *be zero.*

Then the equation (4) can be written as

$$b'\left(y' + \frac{f'}{b'}\right)^2 = -2g'x' + \frac{f'^2 - b'c'}{b'}. \quad \ldots (6)$$

(*a*) If $g' = 0$, then (6) represents *a pair of parallel straight lines* provided $(f'^2 - b'c')$ is positive.

If $g' = 0$ and $(f'^2 - b'c')$ be negative, then the equation (6) does not represent any geometrical locus.

If $g' = 0$ and $f'^2 = b'c'$, then (6) represents *two coincident straight lines.*

(*b*) If $g' \neq 0$, then (6) becomes

$$\left(y' + \frac{f'}{b'}\right)^2 = -\frac{2g'}{b'}\left(x' - \frac{f'^2 - b'c'}{2b'g'}\right).$$

Shifting the origin to the point $\left(\frac{f'^2 - b'c'}{2b'g'}, -\frac{f'}{b'}\right)$ without changing the directions of the axes, this equation becomes

$$Y^2 = -\frac{2g'}{b'}X,$$

where (X, Y) are the new co-ordinates.

This represents a parabola in its canonical form whose axis is the X-axis, that is, whose axis is parallel to the x-axis.

Similarly, if $a' \neq 0$ and $b' = 0$, then the equation will represent a parabola with its axis parallel to the y-axis.

Hence, *in all cases, the general equation of the second degree represents a conic.*

Furthermore, we have seen how a general equation of second degree can be reduced to its canonical form and identify the conic represented by it.

Note. We are familiar with three types of conics, *viz.*, ellipse, hyperbola and parabola as given by the focus-directrix definition. These are the *geometric classes* of conics. We now widen our definition and define a second order curve to be a conic which includes all the cases as discussed before and call them *algebraic classes* of conics.

10.2. Nature of the conic.

The general equation of the second degree in x and y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad \ldots \qquad (1)$$

This represents a second order curve and hence a conic.

Let us introduce the notations

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

and $D = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$.

Δ is called the *discriminant of* (1) and is invariant under translation and rotation of axes.

D is also invariant under translation and rotation of axes.

If, $\Delta = 0$, then the equation (1) represents *a pair of straight lines.* These two lines are *intersecting* if $D \neq 0$ and *parallel* (or *coincident*) if $D = 0$.

If $\Delta = 0$, then the equation (1) is said to represent a *degenerate conic.*

If $a = b$ and $h = 0$, then the equation (1) represents a *circle.*

If $\Delta \neq 0$, then the equation (1) represents either *an ellipse,* or *a hyperbola,* or *a parabola* (*Non-degenerate conic*). These are proper conics.

Let $S(\alpha, \beta)$ be the focus, $lx + my + n = 0$ be the directrix and $P(x, y)$ be a point on the conic whose eccentricity is e. If M be the foot of the perpendicular from P to the directrix, then, by definition of conic,

$$SP = e.PM, \text{ that is, } SP^2 = e^2.PM^2$$

or, $$(x-\alpha)^2 + (y-\beta)^2 = e^2.\frac{(lx+my+n)^2}{l^2+m^2}$$

or, $$\{l^2(1-e^2)+m^2\}x^2 - 2lme^2xy + \{l^2 + m^2(1-e^2)\}y^2$$
$$-2\{(l^2+m^2)\alpha + lne^2\}x - 2\{(l^2+m^2)\beta + mne^2\}y$$
$$+\{(l^2+m^2)(\alpha^2+\beta^2) - n^2e^2 = 0.$$

If this equation represents the equation (1), then we can write that

$a = l^2(1-e^2) + m^2$, $b = l^2 + m^2(1-e^2)$, $h = -lme^2$,

$g = -\{(l^2+m^2)\alpha + lne^2\}$, $f = -\{(l^2+m^2)\beta + mne^2\}$,

$c = (l^2+m^2)(\alpha^2+\beta^2) - n^2e^2$.

Now $D = ab - h^2 = \{l^2(1-e^2)+m^2\}\{l^2+m^2(1-e^2)\} - l^2m^2e^4$

$= (l^2+m^2)^2(1-e^2)$.

For a parabola, we have $e = 1$ and hence $D = 0$.
For an ellipse, we have $e < 1$ and hence $D > 0$.
For a hyperbola, we have $e > 1$ and hence $D < 0$.

Thus an equation of the second degree is said to be of the *parabolic type* if $D = 0$, of the *elliptic type* if $D > 0$ and of the *hyperbolic type* if $D < 0$.

To summarise : the general equation of the second degree will represent

(a) (i) *a pair of straight lines,* if $\Delta = 0$;

(ii) *a pair of intersecting straight lines,*

if $\Delta = 0$ and $D \neq 0$;

(iii) *a pair of parallel* (or *coincident*) *straight lines,*

if $\Delta = 0$ and $D = 0$.

(b) *a circle,* if $a = b$ and $h = 0$.

(c) *a parabola,* if $\Delta \neq 0$ and $D = 0$.

(d) *an ellipse,* if $\Delta \neq 0$ and $D > 0$.

(The ellipse is *real* when $\Delta < 0$ and *imaginary,* that is, there is no geometric locus when $\Delta > 0$.)

(e) (i) *a hyperbola,* if $\Delta \neq 0$ and $D < 0$;

(ii) *a rectangular hyperbola,*

if $\Delta \neq 0$, $D < 0$ and $a + b = 0$.

These results are given in a tabular form below :

D	Δ	Canonical form	Nature
$D > 0$	$\Delta < 0$	$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$	Ellipse
$D > 0$	$\Delta > 0$	$\frac{x^2}{A^2} + \frac{y^2}{B^2} = -1$	Imaginary ellipse
$D < 0$	$\Delta < 0$	$\frac{x^2}{A^2} - \frac{y^2}{B^2} = -1$	Hyperbola
$D < 0$	$\Delta > 0$	$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$	Hyperbola
$D = 0$	$\Delta \neq 0$	$y^2 = 4\alpha x$ or $x^2 = 4\beta y$	Parabola
$D > 0$	$\Delta = 0$	$Ax^2 + By^2 = 0$	Point ellipse or pair of imaginary straight lines
$D \neq 0$	$\Delta = 0$	$y^2 - k^2x^2 = 0$	Pair of intersecting straight lines
$D = 0$	$\Delta = 0$	$y^2 = m^2 (\neq 0)$ or $x^2 = n^2 (\neq 0)$	Pair of parallel straight lines
$D = 0$	$\Delta = 0$	$y^2 = 0$ or $x^2 = 0$	Pair of coincident straight lines

10.3. Conic with centre at the origin.

The point, any chord through which is bisected at that point, is called the *centre* of the conic.

If a non-degenerate conic has a centre, it is called a *central conic*. Otherwise, it is called *a non-central conic*. Thus ellipse (or hyperbola) is a central conic and parabola is a non-central conic.

Let the equation of the central conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If the centre O of the conic be at the origin, then every chord through the centre is bisected at the origin. If POP' be a chord and the co-ordinates of P be (x_1, y_1), then those of P' will be $(-x_1, -y_1)$ and since both the points lie on the conic, we have

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$$

and $$ax_1^2 + 2hx_1y_1 + by_1^2 - 2gx_1 - 2fy_1 + c = 0.$$

Subtracting, we get $4gx_1 + 4fy_1 = 0$.

This is true for all values of x_1 and y_1 on the conic.

Therefore $g = 0$ and $f = 0$.

Hence the equation of the conic having its centre at the origin is

$$ax^2 + 2hxy + by^2 + c = 0.$$

It follows that if the origin be the centre of the conic, then the coefficients of the first degree terms in the equation are all zero.

10.4. Centre of a conic.

Let the equation of the central conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let the centre of the conic be at (x_1, y_1).

Transferring the origin to the point (x_1, y_1) without changing the directions of the axes, the equation becomes

$$a(x' + x_1)^2 + 2h(x' + x_1)(y' + y_1) + b(y' + y_1)^2 + 2g(x' + x_1)$$
$$+ 2f(y' + y_1) + c = 0$$

or, $$ax'^2 + 2hx'y' + by'^2 + 2(ax_1 + hy_1 + g)x'$$
$$+ 2(hx_1 + by_1 + f)y' + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Now the centre being at the origin, the coefficients of x' and y' should separately vanish, that is,

$$ax_1 + hy_1 + g = 0$$

and $$hx_1 + by_1 + f = 0.$$

If now $ab - h^2 \neq 0$, this system of equations is consistent and determinate, that is, it has a unique solution. In that case the co-ordinates of the centre can be found as

$$\frac{x_1}{hf - bg} = \frac{y_1}{gh - af} = \frac{1}{ab - h^2}.$$

Therefore $$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{gh - af}{ab - h^2}.$$

Hence the centre of the conic is given by

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right), \quad ab - h^2 \neq 0.$$

Note 1. If $ab - h^2 = 0$, then the centre is at infinity and the conic, in fact, is a parabola. On the other hand, if $ab - h^2 \neq 0$, then the centre is at finite distance and the conic is either an ellipse or a hyperbola.

Using the notations of capital letters for co-factors in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

the co-ordinates of the centre can be written as

$$\left(\frac{G}{C}, \frac{F}{C}\right), \; C \neq 0.$$

C here has the same expression as D, used earlier.

Note 2. Expressing the equation of the conic as $\phi(x, y) = 0$, the equations determining the centre will be given by

$$\frac{\partial \phi}{\partial x} = 0 \text{ and } \frac{\partial \phi}{\partial y} = 0.$$

Note 3. (*i*) For a central conic, $\Delta \neq 0$ and $D \neq 0$.

The conic in this case is either an ellipse or a hyperbola.

(*ii*) For a conic having no centre (that is, for a non-central conic),

$$D = 0, \; G \neq 0, \; F \neq 0.$$

In this case, the centre is at infinity and the conic is, in fact, a parabola.

(*iii*) For a conic having infinitely many centres,

$$D = 0, \; G = 0, \; F = 0.$$

10.5. Reduction of the equation of a conic.

Case I. *When the conic is central* ($\Delta \neq 0$, $D \neq 0$).

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \ldots \quad (1)$$

whose centre is at the point (x_1, y_1).

Transferring the origin to the centre (x_1, y_1) without changing the directions of the axes, the equation (1) reduces to

$$ax'^2 + 2hx'y' + by'^2 + 2(ax_1 + hy_1 + g)x' + 2(hx_1 + by_1 + f)y'$$
$$+ ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$$

or, $ax'^2 + 2hx'y' + by'^2 + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0,$

where (x', y') are the new co-ordinates and (x_1, y_1) is the centre of the conic so that

$$ax_1 + hy_1 + g = hx_1 + by_1 + f = 0. \quad \ldots \quad (2)$$

The expression

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

$$= x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + gx_1 + fy_1 + c$$

$$= gx_1 + fy_1 + c, \qquad \text{by (2)}$$

$$= g\left(\frac{hf - bg}{ab - h^2}\right) + f\left(\frac{gh - af}{ab - h^2}\right) + c,$$

since (x_1, y_1) is the centre of the conic (1)

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{D} = d\ (\text{say}).$$

Hence the equation (1) becomes

$$ax'^2 + 2hx'y' + by'^2 + d = 0, \qquad \ldots \quad (3)$$

in which the second degree terms remain the same as in the equation (1), first degree terms are absent and c is changed to

$$d = gx_1 + fy_1 + c = \frac{\Delta}{D}.$$

Hence the equation of the conic referred to parallel axes through the centre is of the form

$$Ax^2 + 2Hxy + By^2 = 1.$$

Let us now rotate the axes about the new origin such that the $x'y'$- term is removed. The required angle of rotation will be given by

$$\tan 2\theta = \frac{2h}{a - b}.$$

Now, if the required transformed equation be

$$a'X^2 + b'Y^2 + d = 0, \qquad \ldots \quad (4)$$

we have, by the property of invariants,

$$a + b = a' + b' \text{ and } ab - h^2 = a'b'.$$

These will give a' and b' and hence (4) gives the final transformed equation which is in its *standard* or *canonical* form.

The axes of the conic are $X = 0$ and $Y = 0$,

that is, $(x - x_1)\cos\theta + (y - y_1)\sin\theta = 0$ and

$-(x - x_1)\sin\theta + (y - y_1)\cos\theta = 0$, where $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a - b}$.

Case II. *When the conic is non-central* $(\Delta \neq 0,\ D = 0)$.

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad \ldots \quad (5)$$

This equation will represent a parabola, if $ab-h^2=0$, that is, the terms of the second degree form a perfect square and are of the form

$$(\alpha x+\beta y)^2+2gx+2fy+c=0$$

or, $(\alpha x+\beta y)^2=-2gx-2fy-c$

or, $(\alpha x+\beta y)^2+2(\alpha x+\beta y)\lambda+\lambda^2=2(\alpha\lambda-g)x+2(\beta\lambda-f)y+\lambda^2-c$

or, $(\alpha x+\beta y+\lambda)^2=2(\alpha\lambda-g)x+2(\beta\lambda-f)y+(\lambda^2-c)$. ... (6)

Now we know that in a parabola the axis and the tangent at the vertex are perpendicular to one another and the parabola is the locus of a point which moves such that the square of its distance from the axis varies directly as its distance from the tangent at the vertex.

Now let us choose λ such that the two straight lines

$$\alpha x+\beta y+\lambda=0 \qquad \text{... (7)}$$

and $2(\alpha\lambda-g)x+2(\beta\lambda-f)y+(\lambda^2-c)=0$ ... (8)

are at right angles. Hence

$$\alpha(\alpha\lambda-g)+\beta(\beta\lambda-f)=0, \quad \text{that is,} \quad \lambda=\frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

For this value of λ, the lines (7) and (8) are at right angles.

The straight lines (7) and (8) are respectively the axis and the tangent at the vertex of the parabola for this value of λ. Solving (7) and (8), we get the vertex of the parabola.

Let us now choose these two perpendicular straight lines as co-ordinate axes with reference to which the co-ordinates of a point $P(x,y)$ on the curve are (X,Y).

Then X = the perpendicular distance of $P(x,y)$ from the straight line (8)

$$=\frac{2(\alpha\lambda-g)x+2(\beta\lambda-f)y+\lambda^2-c}{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}} \qquad \text{... (9)}$$

and Y = the perpendicular distance of $P(x,y)$ from the straight line (7)

$$=\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}}. \qquad \text{... (10)}$$

Thus, with reference to the new axes, (6) can be written as

$$(\alpha^2+\beta^2)Y^2=2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}\,X$$

or, $$Y^2=\frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{\alpha^2+\beta^2}X,$$

which is the standard equation of a parabola whose axis is the X-axis for which $Y=0$, that is, $\alpha x+\beta y+\lambda=0$ and the tangent at the vertex is the Y-axis for which $X=0$, that is, $2(\alpha\lambda-g)x+2(\beta\lambda-f)y+\lambda^2-c=0$ with the above value of λ.

The length of the latus rectum is

$$\frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{\alpha^2+\beta^2}=\frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}, \text{ putting for } \lambda.$$

Alternative method.

Using matrices, the equation of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \dots (1)$$

can be written as $X'AX+2B'X+cI_1=O$, $\qquad \dots (2)$

where $A=\begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, $B=\begin{bmatrix} g \\ f \end{bmatrix}$ and dash denotes the transpose of the matrix.

The characteristic equation of the matrix A is

$$\begin{vmatrix} a-\lambda & h \\ h & b-\lambda \end{vmatrix}=0, \text{ that is, } |A-\lambda I|=0.$$

Let λ_1 and λ_2 be the roots of this equation and D_1, D_2 be the corresponding eigen vectors so that $AD_1=\lambda_1 D_1$ and $AD_2=\lambda_2 D_2$.

Let $D_1{}^*$ and $D_2{}^*$ be the unit eigen vectors in the directions of the principal axes of the conic as given by (1); then the orthogonal transformation $X=PX_1$, where $X_1=\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ and $P=[D_1^* \ D_2^*]$ to the equation (2) will reduce it to

$$(PX_1)'APX_1+2B'PX_1+cI_1=O$$

or, $\qquad X_1'(P'AP)X_1+2B'PX_1+cI_1=O. \qquad \dots (3)$

Now A is a symmetric matrix. Hence there exists an orthogonal matrix P such that $P'AP$ is a diagonal matrix with the characteristic roots of A as its diagonal elements. Hence the equation (3) reduces to

$$X_1'DX_1+2B'PX_1+cI_1=O, \quad \text{where } D=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Thus the transformed equation is

$$[\xi \ \ \eta]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}+2[g \ f]\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}+cI_1=O.$$

The reduced equation is thus

$$\lambda_1\xi^2+\lambda_2\eta^2+2(gl_1+fm_1)\xi+2(gl_2+fm_2)\eta+c=0.$$

This is the principal form of the conic.

Then, by a translation of the axes to a suitable origin, we can get rid of the first degree terms in ξ and η and the resulting equation will be the canonical form of the given equation.

10.6. Lengths and position of the axes of a central conic.

Let
$$ax^2+2hxy+by^2=1 \quad \ldots (1)$$
be the equation of a conic referred to its centre as origin.

Let us consider a concentric circle with the conic with radius r as
$$x^2+y^2=r^2. \quad \ldots (2)$$

The equation of the pair of straight lines joining the origin to the points of intersection of (1) and (2) is
$$ax^2+2hxy+by^2=\frac{x^2+y^2}{r^2}, \text{ i.e., } \left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0 \quad \ldots (3)$$

If r be equal to any semi-axis of the conic, then the circle would touch the conic at the end of the axis concerned and hence the straight lines given by (3) would coincide with the axis. Choosing r such that the equation (3) may represent a pair of coincident straight lines, we get the equation
$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)-h^2=0 \ldots (4), \quad \text{or, } \frac{1}{r^4}-(a+b)\frac{1}{r^2}+ab-h^2=0, \ldots (5)$$
which will give the lengths of the axes.

If r_1^2 and r_2^2 be the roots of (5) and if both be positive, then the conic is an ellipse the smaller value r_2^2 corresponds to the square of the semi-minor axis and the greater value r_1^2 corresponds to the square of the semi-major axis. The lengths of the major and minor axes are respectively $2r_1$ and $2r_2$.

If $r_1^2=r_2^2$ and each be positive, then the conic is a circle.

If r_1^2 and r_2^2 have different signs, then the conic is a hyperbola and the positive value corresponds to the square of the semi-transverse axis while the negative value equals the square of the semi-conjugate axis.

The lengths of the axes will be $2r_1$ and $2\sqrt{|r_2^2|}$, where the roots of (4) are r_1^2 and $(-r_2^2)$.

To find the position of the axes, we multiply (3) by $\left(a-\frac{1}{r^2}\right)$ and get
$$\left(a-\frac{1}{r^2}\right)^2x^2+2h\left(a-\frac{1}{r^2}\right)xy+\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)y^2=0, \text{ giving}$$
$$\left(a-\frac{1}{r^2}\right)^2x^2+2h\left(a-\frac{1}{r^2}\right)xy+h^2y^2=0, \text{ by(4), i.e., } \left\{\left(a-\frac{1}{r^2}\right)x+hy\right\}^2=0.$$

This equation represents two coincident straight lines.

The equations of the axes corresponding to r_1^2 and r_2^2 are thus
$\left(a-1/r_1^2\right)x+hy=0$ and $\left(a-1/r_2^2\right)x+hy=0$.

The first gives the equation of the major or transverse axis while the second gives the minor or conjugate axis, referred to the centre as origin. To find the equation of axis referred to $(0,0)$ as origin, we are to replace x by $(x-x')$ and y by $(y-y')$ where (x',y') are the co-ordinates of the centre of the conic.

If r_1^2 be the square of the semi-major axis and r_2^2 be that of the semi-minor axis, then the eccentricity of the conic is given by

$$e^2=(r_1^2-r_2^2)/r_1^2, \text{ that is, } e=\sqrt{1-r_2^2/r_1^2}.$$

The value of r_2^2 must be substituted with its sign.

The foci of a conic lie on the major or transverse axis at equal distances ae from the centre where e is the eccentricity and a is the length of the major or transverse axis. If r_1^2 and r_2^2 be the squares of the semi-axes and θ be the inclination of the major axis to the x-axis, then the co-ordinates of the foci will be $\quad (x'\pm er_1\cos\theta, y'\pm er_1\sin\theta)$,

where the co-ordinates of the centre are (x',y').

The directrices are the straight lines perpendicular to the major or transverse axis at a distance $(\pm r_1/e)$ from the centre. Hence their equations are $(x-x')\cos\theta+(y-y')\sin\theta=\pm\frac{r_1}{e}=\pm r_1^2\Big/\sqrt{r_1^2-r_2^2}$.

The length of the latus rectum is $2r_2^2/r_1$.

10.7. Intersection of a conic and a straight line in a given direction.

Let the equation of the conic be

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad \dots (1)$$

Let a straight line be drawn through (x_1,y_1) making an angle θ with the x-axis whose equation is thus

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \quad \text{that is,} \quad \frac{x-x_1}{l}=\frac{y-y_1}{m}=r, \qquad \dots (2)$$

where we put $\cos\theta=l$ and $\sin\theta=m$.

Now a point on (2), that is $x=x_1+lr, y=y_1+mr$ will lie on the conic (1), if

$$a(x_1+lr)^2+2h(x_1+lr)(y_1+mr)+b(y_1+mr)^2+2g(x_1+lr)+2f(y_1+mr)+c=0$$

that is, $(al^2+bm^2+2hlm)r^2+2r\{(ax_1+hy_1+g)l+(hx_1+by_1+f)m\}$

$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0, \qquad \dots (3)$$

r being the distance of any point on the line from (x_1,y_1).

The roots of the quadratic (3) give the two values of r corresponding to the two points of intersection of (1) and (2).

Hence, in general, a straight line cuts a conic in two points.

10.8. Equation of the tangent to a conic.

The *tangent* to a conic at a point is the limiting position of a straight line when the two points of intersection of the line and the conic coincide at the point.

Let (x_1, y_1) be a point on the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Then $$S_1 \equiv ax_1^2 + 2hx_1 y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \quad \ldots \quad (4)$$

Now (4) shows that one root of (3) (Art. 10.7) is zero.

If both the roots of (3) be zero, that is to say, if (2) be a tangent to (1), then in addition to (4), we have

$$(ax_1 + hy_1 + g)\, l + (hx_1 + by_1 + f)\, m = 0. \quad \ldots \quad (5)$$

Eliminating l and m between (2) and (5), we get the equation of the tangent as

$$(x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f) = 0$$

or, $$axx_1 + h(xy_1 + x_1 y) + byy_1 + gx + fy$$
$$= ax_1^2 + 2hx_1 y_1 + by_1^2 + gx_1 + fy_1$$
$$= -gx_1 - fy_1 - c, \text{ by } (4).$$

Therefore

$$axx_1 + h(xy_1 + x_1 y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

This is the equation of the tangent to $S = 0$ at the point (x_1, y_1) and is generally written for convenience as $T = 0$.

Note. To write T from S we are to replace x^2 by xx_1, y^2 by yy_1, $2xy$ by $(xy_1 + x_1 y)$, $2x$ by $(x + x_1)$ and $2y$ by $(y + y_1)$ and to retain the constant term as it is.

10.9. Condition of tangency of a straight line to a conic.

Let the straight line $$lx + my + n = 0 \quad \ldots \quad (1)$$

be a tangent to the conic $S = 0$ at the point (x_1, y_1).

Then $$T = 0$$

that is, $(ax_1 + hy_1 + g)\, x + (hx_1 + by_1 + f)\, y + (gx_1 + fy_1 + c) = 0$

must be identical with (1).

Comparing the coefficients of x, y and the absolute terms, we have

$$\frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n} = \lambda \text{ (say)}.$$

Then $ax_1 + hy_1 + g - l\lambda = 0$,

$$hx_1 + by_1 + f - m\lambda = 0$$

and $gx_1 + fy_1 + c - n\lambda = 0$.

Also, since (x_1, y_1) lies on (1), we have $lx_1 + my_1 + n = 0$.

Eliminating x_1, y_1 and λ, we get the required condition as

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0$$

or, $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$,

where $A, B, C, \ldots\ldots$ are the co-factors of $a, b, c, \ldots\ldots$ in

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

10.10. Equation of a pair of tangents.

Let the equation of the conic be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The equation of the chord of contact of tangents drawn from the point (x_1, y_1) is

$$T \equiv axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Now the equation $S + \lambda T^2 = 0$,

that is, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$+ \lambda\{axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c\}^2 = 0$$

represents all conics having double contact with the conic $S = 0$ at the ends of the chord $T = 0$. If this conic passes through the point (x_1, y_1), then it becomes the pair of tangents from the point (x_1, y_1). Thus, if (x, y) be replaced by (x_1, y_1), then this becomes

$$S_1 + \lambda S_1^2 = 0.$$

Therefore $\lambda = -\dfrac{1}{S_1}$,

where $S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$.

Hence the equation of the pair of tangents is $SS_1 = T^2$.

10.11. Director circle.

We know that the locus of a point from which a pair of perpendicular tangents can be drawn to a conic is called the *director circle* of the conic.

Let (x_1, y_1) be a point on the director circle of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Then the equation of the pair of tangents from the point (x_1, y_1) to the conic $S = 0$ is $T^2 = SS_1$,

that is,

$$\{ax x_1 + h(xy_1 + x_1 y) + byy_1 + g(x + x_1) + f(y + y_1) + c\}^2$$
$$= (ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$\times (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c).$$

If the two tangents be at right angles as is the case here, then the sum of the coefficients of x^2 and y^2 is zero.

This gives $(ax_1 + hy_1 + g)^2 - aS_1 + (hx_1 + by_1 + f)^2 - bS_1 = 0$

or, $(ax_1 + hy_1 + g)^2 + (hx_1 + by_1 + f)^2$
$$- (a + b)(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) = 0$$

or, $(h^2 - ab)(x_1^2 + y_1^2) + 2(hf - bg)x_1 + 2(gh - af)y_1$
$$+ f^2 + g^2 - bc - ca = 0.$$

Hence the director circle which is the locus of (x_1, y_1) is

$$(h^2 - ab)(x^2 + y^2) + 2(hf - bg)x + 2(gh - af)y$$
$$+ f^2 + g^2 - bc - ca = 0.$$

Note. If the conic be a parabola, then $h^2 = ab$ and the equation reduces to $2(hf - bg)x + 2(gh - af)y + f^2 + g^2 - bc - ca = 0$, which is the directrix of the parabola.

10.12. Equation of the chord of contact of tangents.

We know that the *chord of contact* of tangents from a point (x_1, y_1) is the straight line joining the points of contact of the pair of tangents from the external point (x_1, y_1) to the conic.

Let (x_1, y_1) be a point outside the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

We know that two tangents can be drawn from (x_1, y_1) to the conic and let the points of contact be (x_2, y_2) and (x_3, y_3).

Now the equation of the tangent at (x_2, y_2) to $S = 0$ is

$$axx_2 + h(xy_2 + yx_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

It passes through the point (x_1, y_1). Therefore

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

Similarly, since the tangent at (x_3, y_3) passes through (x_1, y_1), we have

$$ax_1x_3 + h(x_1y_3 + y_1x_3) + by_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0.$$

These two relations show that both the points (x_2, y_2) and (x_3, y_3) lie on the straight line

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

which is the chord of contact of the tangents drawn from the point (x_1, y_1) to the conic $S = 0$.

10.13. Equation of the normal to a conic.

We know that the *normal* to a conic at a point is the straight line which passes through the point and which is perpendicular to the tangent of the conic at that point.

The equation of the tangent at the point (x_1, y_1) to the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0.$$

Its slope is $\left(-\dfrac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}\right)$.

Then the slope of the normal will be $\dfrac{hx_1 + by_1 + f}{ax_1 + hy_1 + g}$.

Hence the equation of the normal to $S = 0$ at (x_1, y_1) is

$$y - y_1 = \frac{hx_1 + by_1 + f}{ax_1 + hy_1 + g}(x - x_1)$$

or, $(x - x_1)(hx_1 + by_1 + f) = (y - y_1)(ax_1 + hy_1 + g)$.

10.14. Equation of the polar with respect to a conic.

We know that the *polar* of a point with respect to a conic is the locus of the point of intersection of the tangents to the conic at the extremities of the chords drawn through that point and the point itself is called the *pole* of the polar.

Let AB be a chord passing through the point $P(x_1, y_1)$ and let the tangents at A and B to the conic $S = 0$ meet in $T(\alpha, \beta)$. The locus of T is the polar.

Now AB is the chord of contact of the tangent drawn from $T(\alpha, \beta)$ to the conic $S = 0$. So its equation is

$$a\alpha x + h(\beta x + \alpha y) + b\beta y + g(x + \alpha) + f(y + \beta) + c = 0.$$

This passes through the point $P(x_1, y_1)$.

Therefore $a\alpha x_1 + h(\beta x_1 + \alpha y_1) + b\beta y_1 + g(x_1 + \alpha)$
$+ f(y_1 + \beta) + c = 0.$

Hence the locus of $T(\alpha, \beta)$ is

$$axx_1 + h(xy_1 + x_1 y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

This is the polar of the point P with respect to the conic $S = 0$.

This is the equation of a straight line and so the polar of a point with respect to a conic is a straight line.

Note. The polar of a point with respect to a conic may also be defined in another way.

Let A, B, C, D be four points on a line such that the segments AC, AB and AD are in H. P. In other words, $\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}$, that is, AB is the harmonic mean between AC and AD. The points C, D are said to be the *harmonic conjugates* of the points A, B and vice-versa.

The locus of the harmonic conjugate of a fixed point P with respect to the two points in which any straight line through P cuts a given conic is called the *polar* of P with respect to the conic and P is called the *pole* of this polar.

The equation of the polar of the point $P(x_1, y_1)$ with respect to the conic $S = 0$ may also be obtained with the help of this definition.

Any straight line through $P(x_1, y_1)$ is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \text{ where } l^2 + m^2 = 1.$$

For the points of intersection of the line and the conic, we have

$$a(x_1 + lr)^2 + 2h(x_1 + lr)(y_1 + mr) + b(y_1 + mr)^2 + 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0$$

or, $r^2(al^2 + 2hlm + bm^2) + 2r\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\} + (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) = 0.$

The roots r_1, r_2 of the quadratic equation in r give the directed distances of the points of intersection from the point (x_1, y_1). By the definition, the directed distance ρ of any point (α, β) on the polar is the harmonic mean between r_1 and r_2.

$$\text{Hence } \frac{2}{\rho} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2}$$

$$= \frac{-2\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}}{ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c}. \quad \ldots (1)$$

Also $\dfrac{\alpha - x_1}{l} = \dfrac{\beta - y_1}{m} = \rho$, whence $l = \dfrac{\alpha - x_1}{\rho}$, $m = \dfrac{\beta - y_1}{\rho}$.

Therefore, from (1), we get

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$
$$= -\{(ax_1 + hy_1 + g)(\alpha - x_1) + (hx_1 + by_1 + f)(\beta - y_1)\}$$

or, $a\alpha x_1 + h(\alpha y_1 + \beta x_1) + b\beta y_1 + g(\alpha + x_1) + f(\beta + y_1) + c = 0.$

Hence the locus of (α, β) is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

This is the equation of the polar of the point P with respect to the conic $S = 0$.

10.15. The pole of a straight line with respect to a conic.

We know that the *pole* of the straight line $lx + my + n = 0$ with respect to the conic $S = 0$ is the point whose polar with respect to the conic $S = 0$ is the straight line $lx + my + n = 0$.

Let the pole of the straight line $lx + my + n = 0$ with respect to the conic $S = 0$ be the point (x_1, y_1).

Now the polar of (x_1, y_1) with respect to the conic $S = 0$ is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

or, $x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$.

This is then identical with the equation $lx + my + n = 0$.

Therefore $\dfrac{ax_1 + hy_1 + g}{l} = \dfrac{hx_1 + by_1 + f}{m} = \dfrac{gx_1 + fy_1 + c}{n}$.

From these, (x_1, y_1), that is, the pole can be determined.

Note. Properties of pole and polar have been discussed in Chapter VI.

10.16. Chords in terms of the middle point.

Let the equation of the conic be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and $\dfrac{x - \alpha}{l} = \dfrac{y - \beta}{m} = r$ be a straight line.

For their points of intersection, we have

$$r^2(al^2 + 2hlm + bm^2) + 2r\{l(a\alpha + h\beta + g) + m(h\alpha + b\beta + f)\}$$
$$+ a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0,$$

r being the distance of any point from the point (α, β).

If (α, β) be the middle point of the chord, then the values of r as given by the above quadratic equation will be equal and opposite, that is, the coefficient of r will be zero. Thus

$$(a\alpha + h\beta + g)l + (h\alpha + b\beta + f)m = 0.$$

Eliminating l and m, we get

$$(a\alpha + h\beta + g)(x - \alpha) + (h\alpha + b\beta + f)(y - \beta) = 0$$

or, $a\alpha x + b\beta y + h(x\beta + y\alpha) + g(x + \alpha) + f(y + \beta) + c$

$$= a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c.$$

This is the equation of the chord in terms of the middle point.

Note. This is symbolically expressed as $T = S'$,

where $T = 0$ is the equation of the tangent to the conic $S = 0$ at the point (α, β) and S' is the expression obtained from S by replacing x, y by α, β respectively.

10.17. Diameters and conjugate diameters.

We know that the locus of the middle points of a system of parallel chords of a conic is called a *diameter* of the conic with respect to the system of parallel chords.

Consider the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and the straight line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = r.$

Let the straight line be a chord of the conic $S = 0$ and (α, β) be the middle point of the chord. Then, from the previous article,

$$(a\alpha + h\beta + g)l + (h\alpha + b\beta + f)m = 0.$$

Thus the locus of the middle points of the chords parallel to the straight line $\dfrac{x}{l} = \dfrac{y}{m}$

is $\quad x(al + hm) + y(hl + bm) + gl + fm = 0,$

which is called the *diameter* of the system of chords parallel to the straight line $\dfrac{x}{l} = \dfrac{y}{m}$ and is a straight line.

If the diameter be parallel to the straight line $\dfrac{x}{l'} = \dfrac{y}{m'}$, then

$$\frac{al + hm}{m'} = -\frac{hl + bm}{l'}$$

or, $\quad all' + h(lm' + l'm) + bmm' = 0.$

The symmetry of the result shows that the diameter bisecting the chords parallel to the straight line

$$\frac{x}{l'} = \frac{y}{m'}$$

is parallel to the straight line $\dfrac{x}{l} = \dfrac{y}{m}$.

Thus, *if a diameter bisects chords parallel to a second diameter, then the second diameter bisects chords parallel to the first.*

Such a pair of diameters is called *conjugate diameters.*

Cor. If the two straight lines $y = m_1 x$ and $y = m_2 x$ be parallel to the conjugate diameters of the conic $S = 0$, then

$$a + h(m_1 + m_2) + bm_1 m_2 = 0.$$

Note. Properties of conjugate diameters have been discussed in Chapter VII.

10.18. Intersection of two conics.

The equation of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

contains six constants. Of these, five are independent as obtained by dividing by any one of them. So a conic in general may be made to satisfy five conditions. If five points on a conic be given, then the equation of the conic can be determined.

But, if four of the points lie on a straight line, then we get an infinite number of conics as the equations are not sufficient to get the values of the five constants.

Two conics in general intersect in four points, real or imaginary; for, eliminating x between $S = 0$ and

$$S_1 \equiv a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

we get an equation of the fourth degree in y which gives four values of y. Correspondingly there are four values of x. Hence there are four points of intersection, of which two or more points may coincide, or two or all of them may be imaginary.

10.19. Conics in general.

The equation of any conic passing through the points of intersection of the two conics $S = 0$ and $S_1 = 0$ is

$$S + \lambda S_1 = 0^* . \qquad \ldots (1)$$

(1) is an equation of second degree and hence is a conic passing through the points common to $S = 0$ and $S_1 = 0$.

Some other given condition determines λ.

Let $U = 0$ and $V = 0$ be the equations of two straight lines whose combined equation is $UV = 0$.

The equation of the conic passing through the points of intersection of $S = 0$ and $UV = 0$ is of the form

$$S + \lambda UV = 0 . \qquad \ldots (2)$$

Let A, B and C, D be the points of intersection of $U = 0$ and $V = 0$ with $S = 0$ respectively. Then $S + \lambda UV = 0$ will represent a conic passing through the points A, B, C, D.

*The general form of the equation of any conic passing through the intersection of the two conics $S = 0$ and $S_1 = 0$ is $\lambda_1 S + \lambda_2 S_1 = 0$, where λ_1 and λ_2 are two arbitrary constants, not simultaneously equal to zero.

If A and B coincide, then $S+\lambda UV=0$ touches the conic $S=0$ at these two conicident points at A.

If the two straight lines be coincident, then the points A and C coincide as also the points B and D coincide.

Then the equation of the conic having *double contact* with the conic $S=0$ at the points cut by the straight line $U=0$ is $S+\lambda U^2=0$.

λ is determined from some other given condition.

If again A, B, C coincide but D remains distinct, then the straight line $U=0$ becomes the tangent at A to the conic $S=0$ and the line AD is the straight line $U \doteq 0$. This is the case of *three point contact.*

Lastly, if A, B, C, D all coincide, then $U=0$ coincides with $V=0$ both being tangents to the conic at A. The conic in this case becomes $S+\lambda U^2=0$. The conic is said to have *four point contact.*

10.20. Illustrative Examples.

Ex. 1. *Reduce the following equations to their canonical forms and determine the nature of the conics represented by them :*

(*i*) $3x^2+2xy+3y^2-16x+20=0$; [K. H. 1998]

(*ii*) $x^2+4xy+y^2-2x+2y+6=0$; [V. H. 1992]

(*iii*) $6x^2-5xy-6y^2+14x+5y+4=0$; [B. H. 1992]

(*iv*) $4x^2+4xy+y^2-12x-6y+5=0$; [C. H. 1981]

(*v*) $3x^2+10xy+3y^2-2x-14y-13=0.$

(*i*) Here $a=3, b=3, h=1, g=-8, f=0$ and $c=20$.

Therefore $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$$=3.3.20+2.0(-8).1-3.0-3.(-8)^2-20.(1)^2=-32\neq 0$$

and $\qquad D=ab-h^2=3.3-1^2=8\neq 0$.

Hence it is a central conic.

Let the centre of the conic be at (α,β).

Transferring the origin to the point (α,β) without changing the directions of the axes, the equation becomes with (x',y') as new coordinates

$$3(x'+\alpha)^2+2(x'+\alpha)(y'+\beta)+3(y'+\beta)^2-16(x'+\alpha)+20=0$$

or, $3x'^2+2x'y'+3y'^2+2x'(3\alpha+\beta-8)+2y'(\alpha+3\beta)$
$$+\left(3\alpha^2+2\alpha\beta+3\beta^2-16\alpha+20\right)=0. \qquad \ldots (1)$$

The origin being at the centre, coefficients of x' and y' should separately vanish, that is,

$3\alpha + \beta - 8 = 0$ and $\alpha + 3\beta = 0$, whence $\alpha = 3$, $\beta = -1$.

Thus the centre of the conic is at $(3, -1)$.

Then the equation (1), with origin at $(3, -1)$, becomes

$$3x'^2 + 2x'y' + 3y'^2 - 4 = 0.$$

To remove the $x'y'$-term, we rotate the axes through an angle θ.

Due to this rotation, the equation reduces to [with new co-ordinates (X, Y)]

$$3(X\cos\theta - Y\sin\theta)^2 + 2(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ 3(X\sin\theta + Y\cos\theta)^2 = 4$$

or, $X^2(3 + \sin 2\theta) + 2XY\cos 2\theta + Y^2(3 - \sin 2\theta) = 4$. . . . (2)

For the removal of XY-term, we require that

$$\cos 2\theta = 0, \text{ that is, } \theta = \tfrac{1}{4}\pi.$$

Then (2) becomes

$$X^2(3 + 1) + Y^2(3 - 1) = 4$$

or,

$$\frac{X^2}{1} + \frac{Y^2}{2} = 1.$$

This is the required reduced canonical form of the given equation.

This represents an ellipse.

(ii) Here $a = b = 1$, $h = 2$, $g = -1$, $f = 1$, $c = 6$.

Therefore $\Delta = 1.1.6 + 2.1.(-1).2 - 1.1^2 - 1(-1)^2 - 6.2^2 = -24 \neq 0$

and $D = 1 - 4 = -3 \neq 0$.

Hence it is a central conic.

Let the centre of the conic be at (α, β).

The equations determining the centre are

$\alpha + 2\beta - 1 = 0$ and $2\alpha + \beta + 1 = 0$, giving $\alpha = -1$, $\beta = 1$.

The reduced equation with the centre at the origin is

$x'^2 + 4x'y' + y'^2 + \dfrac{\Delta}{D} = 0$, that is, $x'^2 + 4x'y' + y'^2 + 8 = 0$.

If the finally reduced equation after rotation be

$$a'X^2 + b'Y^2 + 8 = 0,$$

then, by the theory of invariants, we have

$a' + b' = 1 + 1 = 2$ and $a'b' = 1 - 4 = -3$.

Therefore $a' = -1, 3$ and corresponding $b' = 3, -1$.

Hence the reduced standard form of the given equation is

either $-X^2 + 3Y^2 + 8 = 0$, that is, $\dfrac{X^2}{8} - \dfrac{Y^2}{8/3} = 1$

or $3X^2 - Y^2 + 8 = 0$, that is, $\dfrac{Y^2}{8} - \dfrac{X^2}{8/3} = 1$,

each of which is a hyperbola, the equations being in the canonical form.

Note. The transformation can also be effected by the rotation of axes through an angle θ, given by

$$\tan 2\theta = \frac{2h}{a-b}, \text{ giving } \theta = \frac{\pi}{4} \text{ in this case.}$$

(*iii*) Here $a = 6$, $b = -6$, $h = -\frac{5}{2}$, $g = 7$, $f = \frac{5}{2}$, $c = 4$.

Therefore $\Delta = 6.(-6).4 + 2.\frac{5}{2}.7.(-\frac{5}{2}) - 6.(\frac{5}{2})^2 - (-6)(7)^2 - 4.(-\frac{5}{2})^2 = 0$

and $D = -36 - \frac{25}{4} = -\frac{169}{4} \neq 0$.

Thus the given equation represents a pair of intersecting straight lines.

The equations giving their point of intersection (α, β) are

$12\alpha - 5\beta + 14 = 0$, $-5\alpha - 12\beta + 5 = 0$, whence $\alpha = -\frac{11}{13}$, $\beta = \frac{10}{13}$.

Shifting the origin to the point (α, β), that is, to the point $(-\frac{11}{13}, \frac{10}{13})$, without changing the directions of the axes, the given equation reduces to

$$6x'^2 - 5x'y' - 6y'^2 = 0.$$

Rotating the axes through an angle θ with the same origin, this equation becomes [with new co-ordinates (X, Y)]

$$6(X\cos\theta - Y\sin\theta)^2 - 5(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) - 6(X\sin\theta + Y\cos\theta)^2 = 0$$

or, $(6\cos 2\theta - \frac{5}{2}\sin 2\theta)X^2 - (5\cos 2\theta + 12\sin 2\theta)XY - (6\cos 2\theta - \frac{5}{2}\sin 2\theta)Y^2 = 0.$

To remove the XY-term, we put $5\cos 2\theta + 12\sin 2\theta = 0$

or, $$\frac{\sin 2\theta}{-5} = \frac{\cos 2\theta}{12} = \frac{1}{13}.$$

With these values, the given equation reduces to $X^2 - Y^2 = 0$ which represents two intersecting straight lines

$$X - Y = 0, \quad X + Y = 0.$$

(*iv*) Here $a=4, b=1, h=2, g=-6, f=-3, c=5$.

Therefore $\Delta = 4.1.5+2(-3)(-6)2-4.(-3)^2-1.(-6)^2-5.2^2=0$

and $D=4.1-2^2=0$.

Hence the given equation represents a pair of parallel straight lines.

Rotating the axes through an angle θ with the same origin, the given equation becomes [with new co-ordinates (X, Y)]

$$4(X\cos\theta - Y\sin\theta)^2 + 4(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+(X\sin\theta + Y\cos\theta)^2 - 12(X\cos\theta - Y\sin\theta)$$
$$-6(X\sin\theta + Y\cos\theta)+5=0$$

or, $(\frac{5}{2}+2\sin 2\theta+\frac{3}{2}\cos 2\theta)X^2+(4\cos 2\theta-3\sin 2\theta)XY$

$$+(\frac{5}{2}-2\sin 2\theta-\frac{3}{2}\cos 2\theta)Y^2-2(6\cos\theta+3\sin\theta)X$$
$$+2(6\sin\theta-3\cos\theta)Y+5=0.$$

To remove the XY-term, we put $4\cos 2\theta-3\sin 2\theta=0$

or, $\dfrac{\sin 2\theta}{4}=\dfrac{\cos 2\theta}{3}=\dfrac{1}{5}$.

Therefore $\sin\theta=\dfrac{1}{\sqrt{5}}, \cos\theta=\dfrac{2}{\sqrt{5}}$, taking the positive sign.

With these values, the given equation reduces to

$$5X^2-6\sqrt{5}X+5=0$$

or, $(X-\sqrt{5})\left(X-\dfrac{1}{\sqrt{5}}\right)=0$,

which represents two parallel straight lines

$$X-\sqrt{5}=0,\ X-\frac{1}{\sqrt{5}}=0.$$

(*v*) Here $a=3, b=3, h=5, g=-1, f=-7$ and $c=-13$.

Hence $A=\begin{bmatrix} a & h \\ h & b \end{bmatrix}=\begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$.

The characteristic equation is $|A-\lambda I|=0$

or, $\begin{vmatrix} 3-\lambda & 5 \\ 5 & 3-\lambda \end{vmatrix}=0$.

or, $\lambda^2-6\lambda-16=0$, whence $\lambda=8, -2$.

The characteristic roots are thus 8 and (-2).

For $\lambda = 8$, the eigen vector is $k_1\begin{bmatrix}1\\1\end{bmatrix}$. Therefore $D_1^* = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$.

For $\lambda = -2$, the eigen vector is $k_2\begin{bmatrix}1\\-1\end{bmatrix}$. Therefore $D_2^* = \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$.

The given equation, when transferred to the principal axes, is

$$[\xi \;\; \eta]\begin{bmatrix}8 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}\xi\\\eta\end{bmatrix} + 2[-1 \;\; -7]\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\xi\\\eta\end{bmatrix} - 13I_1 = 0,$$

which is $8\xi^2 - 2\eta^2 - 8\sqrt{2}\,\xi + 6\sqrt{2}\eta - 13 = 0$

or, $8\left(\xi - \frac{1}{\sqrt{2}}\right)^2 - 2\left(\eta - \frac{3}{\sqrt{2}}\right)^2 = 8.$

Now transferring the origin with translation of axes such that $X = \xi - \frac{1}{\sqrt{2}}$, $Y = \eta - \frac{3}{\sqrt{2}}$, (X, Y) being the new co-ordinates, the canonical form of the equation becomes $8X^2 - 2Y^2 = 8$

or, $$\frac{X^2}{1} - \frac{Y^2}{4} = 1.$$

Hence the given equation represents a hyperbola.

Ex. 2. *Discuss the nature of the conics*

(*i*) $4x^2 - 4xy + y^2 + 2x - 26y + 9 = 0.$

(*ii*) $x^2 - 6xy + 9y^2 + 12x - 16y - 8 = 0.$

(*i*) Here $a = 4,\, b = 1,\, h = -2,\, g = 1,\, f = -13,\, c = 9.$

Therefore $\Delta = 4.1.9 + 2(-13).1.(-2) - 4(-13)^2 - 1.1^2 - 9.(-2)^2$

$= -625 \neq 0$

and $D = 4.1 - 2^2 = 0.$

Hence the given equation represents a parabola.

The given equation may be written as

$(2x - y)^2 + 2x - 26y + 9 = 0$

or, $(2x - y + \lambda)^2 = -2x + 26y - 9 + \lambda^2 + 4\lambda x - 2\lambda y$

$= 2(2\lambda - 1)x + 2(13 - \lambda)y + \lambda^2 - 9$, λ being any constant.

λ is so chosen that the two straight lines

$2x-y+\lambda=0$ and $2(2\lambda-1)x+2(13-\lambda)y+\lambda^2-9=0$

be at right angles.

This gives $\quad 2\left\{\dfrac{-(2\lambda-1)}{13-\lambda}\right\}=-1$, that is, $\lambda=3$.

Then the given equation becomes

$$(2x-y+3)^2=10x+20y=10(x+2y)$$

or, $$\frac{(2x-y+3)^2}{2^2+1}=\frac{10}{\sqrt{5}}\left\{\frac{x+2y}{\sqrt{1+2^2}}\right\}.$$

Taking the two perpendicular straight lines

$$x+2y=0 \text{ and } 2x-y+3=0$$

as the axes of co-ordinates, the formulae of transformation are

$$X=\frac{x+2y}{\sqrt{1+2^2}} \text{ and } Y=\frac{2x-y+3}{\sqrt{2^2+1}}$$

and this equation reduces to the canonical form $Y^2=\dfrac{10}{\sqrt{5}}X$.

Thus the given equation represents a parabola whose axis is $2x-y+3=0$, the equation of the tangent at the vertex is $x+2y=0$ and the length of the latus rectum is $2\sqrt{5}$ units.

(*ii*) Here $a=1, b=9, h=-3, g=6, f=-8$ and $c=-8$.

$$A=\begin{bmatrix} a & h \\ h & b \end{bmatrix}=\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}.$$

Therefore $|A-\lambda I|=\begin{vmatrix} 1-\lambda & -3 \\ -3 & 9-\lambda \end{vmatrix}=\lambda^2-10\lambda=0$

is the characteristic equation, whose roots are 0, 10.

For $\lambda=0$, the eigen vector is $k_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Therefore $D_1{}^*=\dfrac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

For $\lambda = 10$, the eigen vector is $k_2\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Therefore $D_2^* = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

The equation, when transferred to the principal axes, becomes

$$[\xi\ \eta]\begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} + 2[6\ \ -8]\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} - 8I_1 = O,$$

which gives $$10\eta^2 + 2\sqrt{10}\,\xi - 6\sqrt{10}\,\eta - 8 = 0$$

or, $$5\left(\eta - \frac{3}{\sqrt{10}}\right)^2 + \sqrt{10}\left(\xi - \frac{17}{2\sqrt{10}}\right) = 0.$$

Transferring the origin with translation of axes such that $X = \xi - \frac{17}{2\sqrt{10}}$ and $Y = \eta - \frac{3}{\sqrt{10}}$, (X, Y) being the new co-ordinates, we have the canonical form of the given equation as $5Y^2 = -\sqrt{10}X$

or, $$Y^2 = -\sqrt{\frac{2}{5}}\,X.$$

Hence the given equation represents a parabola.

Ex.3. *Establish the nature of the following curves and find the centre, semi-axes and eccentricity, if any :*

(*i*) $11x^2 + 4xy + 14y^2 - 26x - 32y + 23 = 0$; [C. H. 2002]

(*ii*) $x^2 + 24xy - 6y^2 + 28x + 36y + 16 = 0$.

(*i*) Here $a = 11, b = 14, c = 23, f = -16, g = -13, h = 2$.

Therefore $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 11.14.23 + 2(-16)(-13).2 - 11.(-16)^2 - 14.(-13)^2 - 23.(2)^2$$

$$= 3542 + 832 - 2816 - 2366 - 92 = -900 < 0$$

and $D = ab - h^2 = 11.14 - 2^2 = 154 - 4 = 150 > 0$.

Hence the conic is an ellipse (a central conic).

Let the centre of the conic be at (α, β).

The equations determining the centre are

$$11\alpha + 2\beta - 13 = 0 \text{ and } \alpha + 7\beta - 8 = 0.$$

Solving these equations, we get (1, 1) as the co-ordinates of the centre.

The equation of the conic referred to the centre as origin is

$$11x^2 + 4xy + 14y^2 + c' = 0, \text{ where } c' = \frac{\Delta}{D} = -6.$$

Therefore the equation of the conic referred to the centre as origin is

$$\frac{1}{6}\left(11x^2 + 4xy + 14y^2\right) = 1.$$

The equation, giving the lenghts of the axes, is

$$\left(\frac{11}{6} - \frac{1}{r^2}\right)\left(\frac{14}{6} - \frac{1}{r^2}\right) = \left(\frac{2}{6}\right)^2$$

or, $25r^4 - 25r^2 + 6 = 0$, giving $r_1^2 = \frac{3}{5}, r_2^2 = \frac{2}{5}$.

The signs of r_1^2 and r_2^2 confirm that the conic is an ellipse.

The equation of the major axis (referred to the centre as origin) is

$$\left(\frac{11}{6} - \frac{5}{3}\right)x + \frac{2}{6}y = 0, \text{ that is, } x + 2y = 0.$$

The equation of the minor axis (referred to the centre as origin) is

$$\left(\frac{11}{6} - \frac{5}{2}\right)x + \frac{2}{6}y = 0, \text{ that is, } 2x - y = 0.$$

Shifting the origin back to (0, 0), we get these equations referred to the origin as $(x-1) + 2(y-1) = 0$, that is, $x + 2y - 3 = 0$,

which is the major axis and its length is $2\sqrt{\frac{3}{5}}$ units.

Similarly, the equation of the minor axis referred to (0, 0) as origin is

$$2(x-1) - (y-1) = 0, \text{ that is, } 2x - y - 1 = 0$$

and its length is $2\sqrt{\frac{2}{5}}$ units.

The eccentricity e is given by $e = \sqrt{1 - \frac{r_2^2}{r_1^2}} = \sqrt{1 - \frac{2}{3}} = \sqrt{\frac{1}{3}}$.

(ii) Here $a = 1,\ b = -6,\ c = 16,\ f = 18,\ g = 14,\ h = 12$.

Therefore

$$\Delta = 1.(-6).16 + 2.18.14.12 - 1.(18)^2 - (-6)(14)^2 - 16.(12)^2$$
$$= -96 + 6048 - 324 + 1176 - 2304 = 4500 > 0$$

and $D = 1.(-6) - (12)^2 = -6 - 144 = -150 < 0$.

The conic is a hyperbola. The equations giving the centre (α, β) are $\alpha + 12\beta + 14 = 0$ and $2\alpha - \beta + 3 = 0$, so that the centre is at $(-2, -1)$. The equation of the conic referred to the centre $(-2, -1)$ as origin is

$$x^2 + 24xy - 6y^2 + \frac{\Delta}{D} = 0$$

or, $x^2 + 24xy - 6y^2 - 30 = 0$

or, $\frac{1}{30}(x^2 + 24xy - 6y^2) = 1$.

The equation giving the lengths of the axes is

$$\left(\frac{1}{30} - \frac{1}{r^2}\right)\left(-\frac{6}{30} - \frac{1}{r^2}\right) = \left(\frac{12}{30}\right)^2$$

or, $(r^2 - 30)(r^2 + 5) = -24r^4$

or, $(r^2 - 3)(r^2 + 2) = 0$, giving $r_1^2 = 3$ and $r_2^2 = -2$.

This confirms that the conic is a hyperbola of eccentricity e given by

$$e = \sqrt{1 - \frac{r_2^2}{r_1^2}} = \sqrt{1 + \frac{2}{3}} = \sqrt{\frac{5}{3}}.$$

Equation of the transverse axis referred to the centre as origin is

$$\left(\frac{1}{30} - \frac{1}{3}\right)x + \frac{12}{30}y = 0, \text{ that is, } 3x - 4y = 0$$

and referred to (0, 0) as origin is

$3(x + 2) - 4(y + 1) = 0$, that is, $3x - 4y + 2 = 0$ and its length is $2\sqrt{3}$.

The equation of the conjugate axis referred to the centre as origin is

$$\left(\frac{1}{30} - \frac{1}{-2}\right)x + \frac{12}{30}y = 0, \text{ that is, } 4x + 3y = 0$$

and referred to (0, 0) as origin is $4(x + 2) + 3(y + 1) = 0$,

that is, $4x + 3y + 11 = 0$ and its length is $2\,|r_2| = 2\sqrt{2}$.

Ex. 4. Find the values of b and g such that the equation

$$4x^2 + 8xy + by^2 + 2gx + 4y + 1 = 0$$

represents (i) a conic without any centre, (ii) a conic having infinitely many centres.

(*i*) In order that the general equation of the second degree represents a conic without any centre, we have

$$ab - h^2 = 0,\ hf - bg \neq 0 \text{ and } gh - af \neq 0.$$

Here $4b - 4^2 = 0,\ 4.2 - bg \neq 0$ and $g.4 - 4.2 \neq 0$.

These give $b = 4$ and $g \neq 2$.

(*ii*) In this case, $ab - h^2 = 0,\ hf - bg = 0,\ gh - af = 0$,

that is, $4b - 4^2 = 0,\ 4.2 - bg = 0,\ g.4 - 4.2 = 0$.

These give $b = 4$ and $g = 2$.

Ex. 5. Find the equation to the conic which passes through the point (1, 1) *and also through the intersections of the conic* $x^2 + 2xy + 5y^2 - 7x - 8y + 6 = 0$ *with the straight lines* $2x - y - 5 = 0$ *and* $3x + y - 11 = 0$.

The equation of the required conic is of the form

$$x^2 + 2xy + 5y^2 - 7x - 8y + 6 + \lambda(2x - y - 5)(3x + y - 11) = 0. \quad \ldots (1)$$

This passes through the point (1, 1).

Therefore $-1 + \lambda(-4)(-7) = 0$, giving $\lambda = \frac{1}{28}$.

Hence, from (1), the required equation of the conic is

$$x^2 + 2xy + 5y^2 - 7x - 8y + 6 + \frac{1}{28}(2x - y - 5)(3x + y - 11) = 0$$

or, $34x^2 + 55xy + 139y^2 - 233x - 218y + 223 = 0$.

Ex. 6. *A circle cuts the parabola* $y^2 = 4ax$ *at the four points* $(at_i^2, 2at_i)$, $(i = 1, 2, 3, 4)$. *Show that* $t_1 + t_2 + t_3 + t_4 = 0$.

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

If it meets the parabola at the point $(at^2, 2at)$, then

$$a^2t^4 + 4a^2t^2 + 2agt^2 + 4aft + c = 0$$

or, $$a^2t^4 + 2at^2(2a + g) + 4aft + c = 0.$$

This is a biquadratic in t, having four roots t_1, t_2, t_3, t_4.
Since the coefficient of t^3 is zero, the sum of the four roots is zero.

In other words, $t_1 + t_2 + t_3 + t_4 = 0$.

Examples X

1. Transform the following equations to canonical forms :

(*i*) $x^2 + 4xy + 4y^2 - 20x + 10y - 50 = 0$. [B. H. 2004]

(*ii*) $3x^2 - 2xy + 3y^2 - 4x - 4y - 12 = 0$.

(*iii*) $x^2 - 6xy + y^2 - 4x - 4y + 12 = 0$.

(*iv*) $3x^2 + 10xy + 3y^2 - 12x - 12y + 4 = 0$.

(*v*) $x^2 + 2xy + y^2 - 4x - 4y + 3 = 0$.

(*vi*) $9x^2 + 24xy + 16y^2 - 126x + 82y - 59 = 0$.

2. (*a*) Reducing the following equations to their canonical forms, determine the nature of the conics represented by them

(*i*) $x^2 + 4xy + 4y^2 + 4x + y - 15 = 0$. [*V. H.* 1993]

(*ii*) $x^2 - 2xy + 2y^2 - 4x - 6y + 3 = 0$. [*C. H. 1980* ; *B. H.* 1991]

(*b*) Reduce the equation

$$x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$$

to its standard form and show that it represents a hyperbola.

[*C. H.* 1992]

3. Reducing the equation

$$4x^2 + 4xy + y^2 - 4x - 2y + a = 0$$

to its canonical form, determine the nature of the conic for different values of *a*. [*V. H. 1992* ; *N. B. H. 1993* ; *C. H. 1994*]

4. Show that the conic given by

(*i*) $3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$

represents a hyperbola ;

(*ii*) $3x^2 - 5xy + 6y^2 + 11x - 17y + 13 = 0$

represents an ellipse ;

(*iii*) $9x^2 + 24xy + 16y^2 + 34x + 62y + 47 = 0$

represents a parabola ;

(*iv*) $x^2 - 4xy + 4y^2 - 12x - 6y - 39 = 0$

represents a parabola.

5. Show that the equation

(*i*) $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$

represents a pair of parallel straight lines.

(*ii*) $20x^2 + 15xy + 9x + 3y + 1 = 0$

represents a pair of intersecting straight lines which are equidistant from the origin. [C. H. 1984]

6. (*a*) Show that the equation

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$$

represents an ellipse whose centre is at (2, 3). [B. H. 1990]

(*b*) Show that the equation

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$$

represents a hyperbola whose centre is at (–2, 2). [B. H. 1987]

7. Discuss the nature of the conic [B. H. 2002]

(*i*) $11x^2 - 4xy + 14y^2 - 58x - 44y + 71 = 0$;

(*ii*) $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$;

(*iii*) $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$;

(*iv*) $17x^2 - 12xy + 8y^2 + 46x + 28y + 17 = 0$.

Find the centre, eccentricity and semi-axes, if any.

8. (*a*) Show that the equation

(*i*) $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

represents a parabola whose latus rectum is 3.

(*ii*) $(a^2 + b^2)(x^2 + y^2) = (ax + by - ab)^2$

represents a parabola of latus rectum $\dfrac{2ab}{\sqrt{a^2 + b^2}}$.

(*b*) Show that the equation

(*i*) $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$

represents a parabola whose axis is the straight line $2x - y - 1 = 0$ and latus rectum is $4/\sqrt{5}$. [C. H. 1993 ; B. H. 1994]

(*ii*) $(3x + 4y)^2 + 8x - 156y - 95 = 0$

represents a parabola whose axis is the straight line $3x + 4y = 12$ and latus rectum is 4.

9. (a) Show that the foci of the conic

$$x^2 - 6xy + y^2 - 2x - 2y + 5 = 0$$

are (1, 1) and (– 2, – 2).

[B. H. 1984]

(b) Show that the foci and the directrices of the conic

$$x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$$

are (– 4, – 4), (– 1, –1), $x + y + 7 = 0$, $x + y + 3 = 0$ respectively.

10. (a) Show that the equation

$$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$$

represents a parabola with the points $(-\frac{1}{5}, \frac{2}{5})$ and (1, 2) as its vertex and focus respectively. Show also that the latus rectum of the parabola is 8 units, the axis is $4x - 3y + 2 = 0$ and the directrix is

$$3x + 4y + 9 = 0.$$

(b) Show that the directrix of the parabola

$$x^2 + 2xy + y^2 - 4x + 8y - 6 = 0 \text{ is } 3x - 3y + 8 = 0.$$

11. (a) Show that the equation

$7x^2 - 2xy + 7y^2 + 22x - 10y + 7 = 0$ represents an ellipse whose centre is $(-\frac{3}{2}, \frac{1}{2})$; axes are $x + y + 1 = 0$, $x - y + 2 = 0$, directrices are $x + y - 3 = 0$, $x + y + 5 = 0$ and foci are (– 1, 1), (– 2, 0).

(b) Reduce the equation $3(x^2 + y^2) + 2xy = 4\sqrt{2}\,(x + y)$ to its canonical form. Name the conic and determine the equations of its axes.

[C. H. 1999; B. H. 2002]

12. (a) Show that the product of the semi-axes of the conic

$x^2 - xy + 2y^2 - 2x - 6y + 7 = 0$ is $2/\sqrt{7}$. [C. H. 1988]

(b) Show that the equation of the axes of the conic

(i) $ax^2 + 2hxy + by^2 = 1$ is $xy\,(a - b) = h\,(x^2 - y^2)$;

(ii) $x^2 - xy - 2y^2 - 2x - 6y + 7 = 0$ is

$$x^2 - 2xy - y^2 + 8y - 8 = 0.$$

13. Show that the equation

$$(a^2 + 1)\,x^2 + 2\,(a + b)\,xy + (b^2 + 1)\,y^2 = c,\ (c > 0),$$

represents an ellipse of area $\dfrac{\pi c}{ab - 1}$.

[T. H. 1992]

14. Show that the conic $2x^2 - 5xy - 3y^2 - x - 4y + 6 = 0$ when referred to the centre as origin becomes

$$2x^2 - 5xy - 3y^2 + 7 = 0.$$

15. Show that the centre of the conic

$$ax^2 + 8xy - 3y^2 - 4x + 2y + c = 0,$$

which passes through the points (0, 0) and (1, 0), is at $\left(\frac{1}{14}, \frac{3}{7}\right)$.

16. Show that the curve

(i) $3x^2 - 4xy - 2y^2 + 3x - 12y - 7 = 0$ has a single centre ;

(ii) $4x^2 - 4xy + y^2 - 6x + 8y + 13 = 0$ has no centre ;

(iii) $4x^2 - 4xy + y^2 - 12x + 6y = 11$ has infinitely many centres.

17. (a) Determine the value of a so that the equation $ax^2 + 6xy + 9y^2 + 3x + 6y - 4 = 0$ may represent (i) a parabola, (ii) a conic having no centre.

(b) Find k so that the equation $kx^2 + 4xy + y^2 - 6x - 2y + 2 = 0$ may represent a point ellipse. [N. B. H. 2007]

(c) Find the values of a and f such that the equation

$$ax^2 - 20xy + 25y^2 - 14x + 2fy - 15 = 0$$

represents (i) a conic without any centre,

(ii) a conic having infinitely many centres.

18. (a) Find the equation of the conic which passes through the point $(-1, -1)$ and also through the intersection of the conic $x^2 + 2xy + 5y^2 + x + 16y + 8 = 0$ with the straight lines $2x - y - 3 = 0$ and $3x + y - 3 = 0$.

(b) Find the equation of the conic which passes through the point (1, 1) and also through the intersection of the two conics $2x^2 - 2xy + y^2 - 6x - 4y + 3 = 0$ and $3x^2 + 10xy + 3y^2 - 14x - 2y - 13 = 0$.

(c) Find the equation of the parabola which passes through the intersection of the two conics $x^2 + 6xy - y^2 + 2x - 3y - 5 = 0$ and $2x^2 - 8xy + 3y^2 + 2y - 1 = 0$.

19. Find the equation of the conic passing through the points (0, 0), (1, 0), (0, − 2), (2, − 1) and (1, − 3).

20. (a) Find the equation of the conic which has its centre at (1, − 3) and which passes through the point (0, − 2) and touches the x-axis at the origin.

(b) Find the equation and the nature of the conic which passes through the point $(-2, 0)$, touches the y-axis at the origin and has its centre at the point $(1, 1)$.

21. Find the equation of the tangent to the conic

$$2x^2 + 3xy - 2y^2 - 7x + y + 3 = 0$$

which is perpendicular to the straight line $2x + 3y + 5 = 0$.

22. (a) If the tangent at any point on the conic

$$(x - y)^2 - 2k(x + y) + k^2 = 0, \quad (k \neq 0)$$

makes intercepts a, b on the co-ordinate axes, then show that $a + b = k$.

(b) Find the condition that the straight line $lx + my = n$ touches the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

23. Determine the points on the conic $x^2 + xy + y^2 = 3$ at which the tangents are parallel to the co-ordinate axes.

24. Find the equation of the normal of the conic $2x^2 + y^2 + x + y = 5$ at the point $(1, 1)$.

25. Show that the equation of the director circle of the conic $11x^2 + 24xy + 4y^2 - 2x + 16y + 11 = 0$ is $x^2 + y^2 + 2x - 2y - 1 = 0$.

26. (a) Prove that the locus of the poles of tangents to the conic $ax^2 + 2hxy + by^2 = 1$ with respect to the conic $a'x^2 + 2h'xy + b'y^2 = 1$ is the conic

$$a(h'x + b'y)^2 - 2h(a'x + h'y)(h'x + b'y) + b(a'x + h'y)^2 = ab - h^2.$$

(b) If the chord of contact of tangents from a point to the conic $ax^2 + 2hxy + by^2 = 1$ subtends a right angle at the centre, then prove that the locus of the point is given by

$$(a^2 + h^2)x^2 + 2(a + b)hxy + (b^2 + h^2)y^2 = a + b. \quad \text{[N. B. H. 1991]}$$

(c) Show that the equation of the polar of the origin with respect to the conic $\left(\frac{x}{a} + \frac{y}{c} - 1\right)\left(\frac{x}{b} + \frac{y}{d} - 1\right) + \lambda xy = 0$ is

$$x\left(\frac{1}{a} + \frac{1}{b}\right) + y\left(\frac{1}{c} + \frac{1}{d}\right) - 2 = 0. \quad \text{[K. H. 2008]}$$

(d) Find the condition that the two straight lines $lx + my = 1$ and $px + qy = 1$ will be conjugate with respect to the conic

$$ax^2 + 2hxy + by^2 = 1. \quad \text{[C. H. 2010]}$$

27. (a) Show that the equation to the common conjugate diameters of the conics $x^2 + 4xy + 6y^2 = 1$ and $2x^2 + 6xy + 9y^2 = 1$ is

$$x(x + 3y) = 0.$$

(b) Show that the two straight lines given by the equation $Ax^2 + 2Hxy + By^2 = 0$ may be conjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$, if $aB + bA = 2hH$.

(c) Prove that the equation to the equi-conjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$ is

$$(a + b)(ax^2 + 2hxy + by^2) = 2(ab - h^2)(x^2 + y^2).$$

28. Find the equation of the hyperbola whose asymptotes are the straight lines $x + y - 1 = 0$ and $y - x + 2 = 0$ and which passes through the point $P(1, 1)$. Prove that the eccentricity of this hyperbola is $\sqrt{2}$.

29. (a) Show that $\dfrac{1}{x+y-a} + \dfrac{1}{x-y+a} + \dfrac{1}{-x+y+a} = 0$ represents a parabola whose focus is $\left(\frac{a}{2}, \frac{a}{2}\right)$ and directrix is $x + y = 0$.

(b) The axes being rectangular, prove that the locus of the focus of the parabola $\left(\dfrac{x}{a} + \dfrac{y}{b} - 1\right)^2 = \dfrac{4xy}{ab}$, a and b being variables such that $ab = c^2$, is the curve $(x^2 + y^2)^2 = c^2xy$. [V.H. 1999; N.B.H. 1999]

30. Find the focus, axis and directrix of the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

31. Show that the curve, given by the equations

$$x = at^2 + bt + c, \; y = a't^2 + b't + c' \quad (t \text{ being a parameter}),$$

is a parabola, whose latus rectum is $\dfrac{(ab' - a'b)^2}{(a^2 + a'^2)^{3/2}}$.

32. Show that the conic

$$ax^2 + 2hxy + by^2 + 2gx\sin^2\theta + 2fy\cos^2\theta + c = 0,$$

where θ is a parameter, always passes through two fixed points.

33. If S be the focus and P, Q be two points on a conic such that the angle PSQ is constant and equal to δ, then prove that the locus of the intersection of the tangents at P and Q is a conic whose focus is S.

34. (a) If α, β, γ, δ be the eccentric angles of four points on an ellipse such that the normals at them are concurrent, then show that

$$\alpha + \beta + \gamma + \delta = \text{an odd multiple of } \pi.$$

(b) If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four points in which a circle cuts an ellipse, then show that

$$\alpha + \beta + \gamma + \delta = \text{an even multiple of } \pi.$$

[Let the equation of the ellipse be $S = x^2/a^2 + y^2/b^2 - 1 = 0$. Obtain the equation of the chords joining the points α, β and γ, δ. Call them $U = 0$ and $V = 0$ respectively. Then choose λ such that the equation $S + \lambda UV = 0$ represents a circle through the four points $\alpha, \beta, \gamma, \delta$. For this, the coefficient of xy will be zero.]

35. If the major axes of two conics be parallel, then prove that the four points in which they meet are concyclic.

36. (a) Prove that all conics through the points of intersection of two rectangular hyperbolas are also rectangular hyperbolas.

(b) Show that two parabolas can be drawn through the four points of intersection of two conics.

[If the two conics be $S_1 = 0$ and $S_2 = 0$, then the equation of a conic passing through the intersection of these two conics is $S_1 + \lambda S_2 = 0$. If this represents a parabola, then the terms of second degree must form a perfect square, that is, $(h_1 + \lambda h_2)^2 = (a_1 + \lambda a_2)(b_1 + \lambda b_2)$. This, being a quadratic in λ, gives two values of λ.]

Answers

1. (i) $Y^2 = 2\sqrt{5}X$. (ii) $\frac{X^2}{8} + \frac{Y^2}{4} = 1$. (iii) $\frac{X^2}{8} - \frac{Y^2}{4} = 1$.

(iv) $\frac{X^2}{\frac{5}{8}} - \frac{Y^2}{\frac{5}{2}} = 1$. (v) $2X = 3\sqrt{2}$, $2X = \sqrt{2}$. (vi) $Y^2 = 6X$.

2. (a) (i) Parabola. (ii) Ellipse. (b) $\frac{X^2}{\frac{2}{7}} - \frac{Y^2}{\frac{2}{3}} = 1$ or $\frac{Y^2}{\frac{2}{7}} - \frac{X^2}{\frac{2}{3}} = 1$.

3. Pair of parallel straight lines for $a < 1$;

Pair of coincident straight lines for $a = 1$.

If $a > 1$, then the straight lines become imaginary.

7. (i) Ellipse ; (3, 2) ; $\frac{1}{\sqrt{3}}$; $\sqrt{6}$, 2. (ii) Hyperbola ; (– 2, 2) ; $\sqrt{\frac{6}{5}}$; $\sqrt{2}$, $\sqrt{\frac{1}{5}}$

(iii) Hyperbola ; (– 4, 0) ; $\sqrt{\frac{10}{3}}$; $\frac{1}{7}\sqrt{14}$, $\frac{1}{3}\sqrt{6}$.

(iv) Ellipse, (– 1, 1) ; $\sqrt{\frac{3}{4}}$; 2, 1.

11. (b) $2X^2 + Y^2 = 2$; Ellipse; $x - y = 0$, $x + y = \sqrt{2}$.

17. (a) (i) $a = 1$. (ii) $a = 1$. (b) $k = 5$. (c) (i) $a = 4$, $f = \frac{35}{2}$. (ii) $a = 4$, $f = \frac{35}{2}$

18. (a) $34x^2 + 55xy + 139y^2 + 13x + 348y + 233 = 0$.

(b) $8x^2 - 86xy - 5y^2 + 6x - 40y + 117 = 0$.

(c) $4x^2 + 4xy + y^2 + 4x - 4y - 11 = 0$ or $5x^2 - 10xy + 5y^2 + 2x + y - 7 = 0$.

19. $3x^2 + 2xy + 2y^2 - 3x + 4y = 0$.

20. (a) $6x^2 + 4xy + y^2 + 2y = 0$.

(b) $x^2 - 4xy + 2y^2 + 2x = 0$; Hyperbola.

21. $3x - 2y = 1$.

22. (b) $l^2(bc - f^2) + m^2(ca - g^2) + n^2(ab - h^2) + 2mn(gh - af)$

$+ 2nl(hf - bg) + 2lm(fg - ch) = 0$.

23. $(1, -2), (-1, 2), (2, -1), (-2, 1)$.

24. $3x - 5y + 2 = 0$. 26. (d) $amq - h(pm + ql) + bpl = ab - h^2$.

28. $x^2 - y^2 - 3x - y + 4 = 0$.

30. $\left(\frac{a}{2}, \frac{a}{2}\right)$; $x - y = 0$; $x + y = 0$.

5 TANGENTS AND NORMALS

5.1. Tangents.

Let two points P and Q be taken on any curve and let the point Q move along the curve nearer and nearer to the point P; then the limiting position of the straight line PQ, when Q moves up to and ultimately coincides with P, is called the *tangent* to the curve at the point P. More precisely, the tangent to a conic at a point is the limiting position of a straight line when the two points of intersection of the line and the conic coincide at the point.

5.2. Equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ at the point } (x_1, y_1).$$

Let $P(x_1, y_1)$ be a point on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad \ldots \quad (1)$$

Let Q be another point (x_2, y_2) on the circle very near to P.

The equation of the straight line PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad \ldots \quad (2)$$

Since both P and Q lie on the circle, we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0, \qquad \ldots \quad (3)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0. \qquad \ldots \quad (4)$$

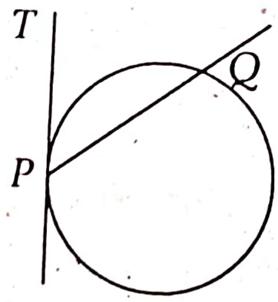

Fig. 20

Subtracting (3) from (4), we get

$$x_2^2 - x_1^2 + y_2^2 - y_1^2 + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

or, $(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0.$

Therefore $\dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}$.

Substituting this in (2), the equation becomes

$$y - y_1 = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}(x - x_1).$$

Hence the equation of the chord PQ is

$$(y - y_1)(y_1 + y_2 + 2f) + (x - x_1)(x_1 + x_2 + 2g) = 0. \quad \ldots \quad (5)$$

As Q approaches and coincides with P, the chord PQ becomes the tangent at P.

Substituting x_1 for x_2 and y_1 for y_2 in (5), we get the equation of the tangent at $P(x_1, y_1)$ as

$$(y - y_1)(2y_1 + 2f) + (x - x_1)(2x_1 + 2g) = 0$$

or, $x(x_1 + g) + y(y_1 + f) = x_1^2 + y_1^2 + gx_1 + fy_1$.

Adding $(gx_1 + fy_1 + c)$ to both sides and using (3), we get the equation of the tangent as

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Second method.

The equations of the straight line through the point (x_1, y_1) making an angle θ with the positive direction of x-axis in the distance form are

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

where r denotes the distance of any point (x, y) on the straight line from the point (x_1, y_1).

Putting $\cos\theta = l$ and $\sin\theta = m$, we rewrite the equations in the form

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \quad \ldots \quad (1)$$

where $l^2 + m^2 = \cos^2\theta + \sin^2\theta = 1$.

If this straight line meets the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \quad (2)$$

at a distance r from (x_1, y_1), the co-ordinates of the meeting point are $(x_1 + lr, y_1 + mr)$ which must satisfy the equation (2).

Therefore

$$(x_1 + lr)^2 + (y_1 + mr)^2 + 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0$$

or, $r^2 + 2r\{l(x_1 + g) + m(y_1 + f)\} + (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = 0,$ ... (3)

since $l^2 + m^2 = 1$.

This is a quadratic in r having two roots r_1 and r_2, which give the directed distances of the points of intersection from (x_1, y_1). If the straight line (1) touches the circle (2) at (x_1, y_1), then each of the two roots of the equation (3) will be zero. Hence, for tangency,

$$r_1 + r_2 = 0 \text{ and } r_1 r_2 = 0.$$

Therefore $l(x_1 + g) + m(y_1 + f) = 0$ ... (4)

and $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$. ... (5)

Eliminating l and m from (1) and (4), we get the equation of the tangent as

$$(x - x_1)(x_1 + g) = -(y - y_1)(y_1 + f)$$

or, $xx_1 + yy_1 + gx + fy - (x_1^2 + y_1^2 + gx_1 + fy_1) = 0$

or, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0,$ by (5).

Cor. The equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) is thus $xx_1 + yy_1 = a^2$.

5.3. Equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two very close points on the parabola $y^2 = 4ax$. ... (1)

The equation of the straight line PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \quad \ldots (2)$$

The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the parabola.

Hence $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$. ... (3)

Therefore $y_2^2 - y_1^2 = 4a(x_2 - x_1)$

or, $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}.$$

Substituting in (2), the equation of the chord PQ becomes

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1).$$

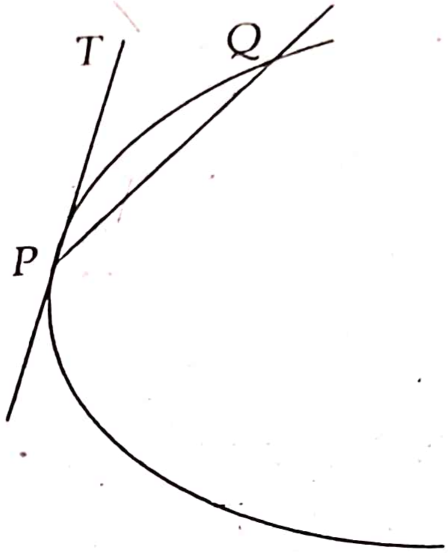

Fig. 21

As Q approaches and coincides with P, the chord PQ becomes the tangent at P.

Substituting x_1 for x_2 and y_1 for y_2, the equation of the tangent becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1)$$

or, $(y - y_1)y_1 = 2a(x - x_1)$

or, $yy_1 - 2ax = y_1^2 - 2ax_1$.

Adding $(-2ax_1)$ to both sides, we get

$$\therefore yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 = 0, \text{ by (3)}.$$

Therefore the equation of the tangent at $P(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1).$$

Cor. The tangent at (0, 0) to the parabola $y^2 = 4ax$ is $x = 0$, that is, the tangent at the vertex is perpendicular to the axis of the parabola. The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$x - ty + at^2 = 0.$$

Note. The equation of the tangent may also be obtained by the second method of Art. 5.2.

5.4. Equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two very close points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad \ldots (1)$$

The equation of the straight line PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \quad \ldots (2)$$

The points $P(x_1, y_1')$ and $Q(x_2, y_2)$ lie on the ellipse.

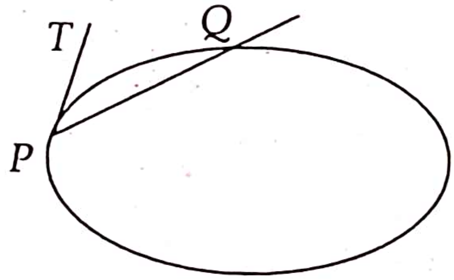

Fig. 22

Hence $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$.

and $\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1. \quad \ldots (3)$

Subtracting, $\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0$

or, $\frac{y_2-y_1}{x_2-x_1}=-\frac{b^2(x_2+x_1)}{a^2(y_2+y_1)}$.

Substituting this in (2), the equation of the chord PQ of the ellipse becomes

$$y-y_1=-\frac{b^2(x_2+x_1)}{a^2(y_2+y_1)}(x-x_1). \quad \ldots (4)$$

As Q approaches and coincides with P, the chord PQ becomes the tangent at P. Substituting x_1 for x_2 and y_1 for y_2 in (4), the equation of the tangent at P becomes

$$y-y_1=-\frac{2b^2x_1}{2a^2y_1}(x-x_1).$$

or, $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad \text{by (3)}.$$

Therefore the equation of the tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Cor. The tangents at the extremities of the major axis, that is, at $(a, 0)$ and $(-a, 0)$ are $x = a$ and $x = -a$ respectively.

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

Note. The equation of the tangent may also be obtained by the second method of Art. 5.2.

5.5. Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

Proceeding exactly in the same way as in the case of ellipse, we get the equation of the tangent to the hyperbola at the point (x_1, y_1) as $$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\phi, b\tan\phi)$ is $\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1.$

Cor. The equation of the tangent to the equilateral hyperbola $x^2 - y^2 = a^2$ at the point (x_1, y_1) is $xx_1 - yy_1 = a^2$.

Note. The equation of the tangent to any conic at the point (x_1, y_1) is obtained by putting xx_1 for x^2, yy_1 for y^2, $(xy_1 + x_1y)$ for $2xy$, $(x + x_1)$ for $2x$, $(y + y_1)$ for $2y$ and retaining the constant term as it is.

5.6. Condition of tangency of a straight line to a circle.

Let the equation of the circle be $x^2 + y^2 = a^2$ ... (1)

and, that of the straight line be $y = mx + c$. ... (2)

The co-ordinates of the points of intersection of the circle and the line are obtained by solving the two equations.

Eliminating y between the two equations, we get

$$x^2 + (mx + c)^2 = a^2$$

or, $$(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0.$$

This quadratic in x gives two roots which are the abscissae of the two points of intersection. If the line be a tangent to the circle, then it cuts the circle at two coincident points and the above equation should give two equal roots. The condition for that is

$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

or, $$c^2 = a^2(1 + m^2)$$

or, $$c = \pm a\sqrt{1 + m^2}.$$

This is the required condition of tangency.

Thus there are two tangents to a circle parallel to a given direction.

Second method.

Let the straight line $y = mx + c$ ... (1)

touch the circle $x^2 + y^2 = a^2$ ... (2)

at the point (x_1, y_1).

The equation of the tangent at the point (x_1, y_1) to the circle (2) is

$$xx_1 + yy_1 = a^2. \qquad \ldots (3)$$

Therefore the equations (1) and (3) are identical, that is, the coefficients of similar terms are proportional.

Hence $\frac{x_1}{m} = \frac{y_1}{-1} = -\frac{a^2}{c}$, giving $x_1 = -\frac{a^2 m}{c}$ and $y_1 = \frac{a^2}{c}$.

Since (x_1, y_1) lies on the circle (2), therefore

$$x_1^2 + y_1^2 = a^2$$

or, $\frac{a^4 m^2}{c^2} + \frac{a^4}{c^2} = a^2$, whence $c = \pm a\sqrt{1 + m^2}$.

This is the required condition of tangency and the point of contact is

$$\left(-\frac{a^2 m}{c}, \frac{a^2}{c}\right).$$

Note. This condition may also be obtained by using the property that the radius of the circle is equal to the distance of the centre of the circle from the tangent line.

5.7. Condition of tangency of a straight line to a parabola.

The points of intersection of the straight line

$$y = mx + c \qquad \ldots \qquad (1)$$

with the parabola

$$y^2 = 4ax \qquad \ldots \qquad (2)$$

are obtained by solving these two equations.

Eliminating y from (1) and (2), we get

$$(mx + c)^2 = 4ax$$

or,

$$m^2x^2 + (2mc - 4a)x + c^2 = 0 . \qquad \ldots \qquad (3)$$

The roots of (3) are the abscissae of the two points of intersection.

If the line (1) touches the parabola (2), then the two points of intersection will be coincident, that is, the roots of (3) are equal.

The condition for equal roots is

$$(2mc - 4a)^2 = 4m^2c^2$$

or, $16a^2 - 16amc = 0$

or, $c = a/m.$

Thus, for any value of $m \neq 0$, the straight line

$$y = mx + a/m$$

touches the parabola $y^2 = 4ax.$

The point of contact can easily be seen to be at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Note. This condition may also be obtained by the second method of Art. 5.6.

5.8. Condition of tangency of a straight line to an ellipse.

The points of intersection of the straight line

$$y = mx + c \qquad \ldots \qquad (1)$$

with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots \qquad (2)$$

are obtained by solving these two equations.

Eliminating y between these two equations, we get

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

or, $$b^2 x^2 + a^2 (mx + c)^2 = a^2 b^2$$

or, $$(a^2 m^2 + b^2) x^2 + 2a^2 mcx + a^2 (c^2 - b^2) = 0.$$

The roots of this equation are the abscissæ of the two points of intersection. If the line be a tangent, then the two points are coincident and hence the roots of this quadratic will be equal. The condition for equal roots is

$$4m^2c^2a^4 - 4(a^2m^2 + b^2)(a^2c^2 - a^2b^2) = 0$$

or, $$a^2m^2c^2 - (a^2m^2c^2 - a^2m^2b^2 + b^2c^2 - b^4) = 0$$

or, $$c^2 = a^2m^2 + b^2.$$

Therefore $$c = \pm\sqrt{a^2m^2 + b^2}.$$

Thus the lines $y = mx \pm \sqrt{a^2m^2 + b^2}$ are tangents to the ellipse for any value of m. Hence there are two tangents to an ellipse parallel to any given direction (that is, for same value of m).

The point of contact of this tangent can easily be verified to be

$$\left(\frac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right).$$

Note. This result may also be obtained by the second method of Art. 5.6.

5.9. Condition of tangency of a straight line to a hyperbola.

Proceeding exactly in the same way as in the case of ellipse, we see that the condition of tangency of the straight line $y = mx + c$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$c = \pm\sqrt{a^2m^2 - b^2}.$$

5.10. Two theorems.

(*a*) *Two tangents can be drawn from a point to a parabola. If these two tangents be perpendicular to one another, the locus of their point of intersection is the directrix.*

The straight line $y = mx + a/m$ always touches the parabola $y^2 = 4ax$. If this line passes through a given point (α, β), we have

$$\beta = m\alpha + a/m$$

or, $$m^2\alpha - m\beta + a = 0,$$

which, being a quadratic in m, has two roots m_1 and m_2 (say). Hence two tangents can be drawn from a point (α, β) to the parabola.

The equations of the tangents are thus

$y = m_1 x + a/m_1$ and $y = m_2 x + a/m_2$, where $m_1 m_2 = a/\alpha$.

If these two tangents be perpendicular to one another, then we have

$$m_1 m_2 = -1$$

or, $a/\alpha = -1$, whence $\alpha + a = 0$.

Hence the locus of (α, β) is the straight line $x + a = 0$, which is the directrix of the given parabola.

(b) *Two tangents can be drawn from a point to a central conic. If these two tangents be perpendicular to one another, then the locus of their point of intersection is a circle.*

The straight line $y = mx + \sqrt{\dfrac{m^2}{a} + \dfrac{1}{b}}$ always touches the central conic $ax^2 + by^2 = 1$. If this line passes through a given point (α, β), then we have

$$\beta = m\alpha + \sqrt{\frac{m^2}{a} + \frac{1}{b}}$$

or, $$m^2\left(\frac{1}{a} - \alpha^2\right) + 2m\alpha\beta + \left(\frac{1}{b} - \beta^2\right) = 0,$$

which, being a quadratic in m, has two roots m_1 and m_2 (say). Hence two tangents can be drawn from a point (α, β) to the central conic.

The equations of the tangents are

$$y = m_1 x + \sqrt{\frac{m_1^2}{a} + \frac{1}{b}} \text{ and } y = m_2 x + \sqrt{\frac{m_2^2}{a} + \frac{1}{b}},$$

where $m_1 m_2 = \left(\dfrac{1}{b} - \beta^2\right) \Big/ \left(\dfrac{1}{a} - \alpha^2\right)$.

If these two tangents be perpendicular to one another, then we have

$$m_1 m_2 = -1, \text{ giving } \frac{1}{b} - \beta^2 = -\left(\frac{1}{a} - \alpha^2\right).$$

Therefore $\alpha^2 + \beta^2 = \frac{1}{a} + \frac{1}{b}$.

Hence the locus of (α, β) is $x^2 + y^2 = \frac{1}{a} + \frac{1}{b}$, which is a circle.

This is called the *director circle* of the central conic.

Cor. The director circle of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $x^2 + y^2 = a^2 + b^2$. The director circle of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $x^2 + y^2 = a^2 - b^2$. In this case, if $a^2 < b^2$, the circle does not exist and if $a^2 = b^2$, that is, if the hyperbola be equilateral, the circle becomes a point circle.

5.11. Length of a tangent to a circle.

Let $P(x_1, y_1)$ be a point outside the circle $x^2 + y^2 = a^2$, whose centre is at $C(0, 0)$.

Let PT be a tangent and CT be the radius a of the circle. Then, in the right-angled triangle CPT, the angle PTC is a right angle.

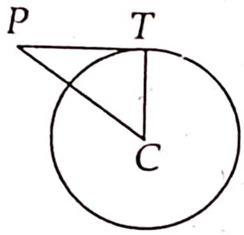

Fig. 23

Hence $PT^2 = PC^2 - TC^2 = x_1^2 + y_1^2 - a^2$.

Thus the length of the tangent is

$$PT = (x_1^2 + y_1^2 - a^2)^{1/2}.$$

If the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$, the co-ordinates of the centre C are $(-g, -f)$ and the radius CT of the circle is $\sqrt{g^2 + f^2 - c}$.

The relation $PT^2 = PC^2 - TC^2$ gives

$$PT^2 = \{x_1 - (-g)\}^2 + \{y_1 - (-f)\}^2 - (g^2 + f^2 - c).$$

Therefore $PT = (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)^{1/2}$.

Second Method.

If the straight line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = r$ be the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ through the point (x_1, y_1) (not lying on the circle), then the two roots r_1 and r_2 of the quadratic equation

$$r^2 + 2r\{l(x_1 + g) + m(y_1 + f)\} + (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = 0$$

(*cf.* Second method, Art. 5.2)

obey the relation $|r_1| . |r_2| = T^2$, where T represents the length of the tangent from the point (x_1, y_1) to the circle.

Since $r_1 r_2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$,

therefore $T^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Note. In both the cases we notice that the square of the length of the tangent drawn from a point to a circle is obtained by replacing x, y by the co-ordinates of the point under consideration, provided the coefficients of x^2 and y^2 in the equation of the circle are each unity.

5.12. Equation of the pair of tangents from an external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$.

Let (h, k) be any point on either of the tangents from the point (x_1, y_1) to the circle $x^2 + y^2 = a^2$.

Now, the length of the perpendicular from the centre to the tangent is equal to the radius of the circle. Here the length of the perpendicular from the origin to the line joining (h, k) and (x_1, y_1) must be equal to a, the radius of the circle. The equation of the straight line passing through (h, k) and (x_1, y_1) is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

or, $$y(h - x_1) - x(k - y_1) + kx_1 - hy_1 = 0. \quad \ldots (1)$$

Hence the length of the perpendicular from the origin to (1) is

$$\frac{kx_1 - hy_1}{\sqrt{(h - x_1)^2 + (k - y_1)^2}} = a$$

or, $(kx_1 - hy_1)^2 = a^2\{(h - x_1)^2 + (k - y_1)^2\}$.

Hence the locus of (h, k) is

$$(x_1 y - xy_1)^2 = a^2\{(x - x_1)^2 + (y - y_1)^2\},$$

which is the required pair of tangents.

This may be simplified as

$$x^2(y_1^2 - a^2) + y^2(x_1^2 - a^2) - a^2(x_1^2 + y_1^2)$$
$$= 2x_1y_1xy - 2a^2x_1x - 2a^2y_1y$$

or, $$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2)$$
$$= x^2x_1^2 + y^2y_1^2 + a^4 + 2x_1y_1xy - 2a^2x_1x - 2a^2y_1y$$
$$= (xx_1 + yy_1 - a^2)^2.$$

Let $S \equiv x^2 + y^2 - a^2$, $S_1 \equiv x_1^2 + y_1^2 - a^2$ and $T \equiv xx_1 + yy_1 - a^2$.

Then the equation of the pair of tangents from the point (x_1, y_1) to the circle is $\quad SS_1 = T^2.$

Second method.

For the points of intersection of the circle $x^2 + y^2 = a^2$ and the straight line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \quad l^2 + m^2 = 1$$

through the point (x_1, y_1), we have

$$(x_1 + lr)^2 + (y_1 + mr)^2 = a^2$$

or, $$r^2 + 2r(lx_1 + my_1) + (x_1^2 + y_1^2 - a^2) = 0.$$

If the straight line be a tangent from the point (x_1, y_1) to the circle, then the two points of intersection will coincide and hence the roots of this quadratic equation in r must be equal.

Therefore $\quad (lx_1 + my_1)^2 = x_1^2 + y_1^2 - a^2.$

Eliminating l, m from the equation of the straight line and the last equation, we have

$$\{x_1(x - x_1) + y_1(y - y_1)\}^2 = (x_1^2 + y_1^2 - a^2)\{(x - x_1)^2 + (y - y_1)^2\}$$

or, $$(xx_1 + yy_1 - a^2)^2 = (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2).$$

This, being satisfied by any point on the tangent drawn through the point (x_1, y_1), gives the equation of the pair of tangents from the point (x_1, y_1) to the circle.

Cor. Proceeding in the same way, we obtain the equation of the pair of tangents from (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ as } SS_1 = T^2,$$

where $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c,$

$\quad S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

and $\quad T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$

5.13. Equation of the pair of tangents from an external point (x_1, y_1) to the parabola $y^2 = 4ax$.

For the points of intersection of the parabola $y^2 = 4ax$ and the straight line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \quad l^2 + m^2 = 1$$

through the point (x_1, y_1), we have

$$(y_1 + mr)^2 = 4a(x_1 + lr)$$

or, $$m^2r^2 - 2r(2al - my_1) + (y_1^2 - 4ax_1) = 0.$$

If the straight line be a tangent from (x_1, y_1) to the parabola, then the two points of intersection will coincide and hence the roots of this quadratic equation in r must be equal.

Therefore $(2al - my_1)^2 = m^2(y_1^2 - 4ax_1)$

or, $$al^2 - lmy_1 + m^2x_1 = 0.$$

Eliminating l, m from the equations of the straight line and the last equation, we have

$$a(x - x_1)^2 - (x - x_1)(y - y_1)y_1 + (y - y_1)^2x_1 = 0$$

or, $$(y^2 - 4ax)(y_1^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2$$

This, being satisfied by any point on the tangent drawn through the point $(x_1; y_1)$, gives the equation of the pair of tangents from the point (x_1, y_1) to the parabola.

Thus the equation of the pair of tangents from the point (x_1, y_1) to the parabola is $SS_1 = T^2$,

where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1^2 - 4ax_1$

and $T \equiv yy_1 - 2a(x + x_1)$.

5.14. Equation of the pair of tangents from an external point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equations of a straight line through the point (x_1, y_1) are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \text{ where } l^2 + m^2 = 1.$$

For the intersection of this straight line and the ellipse, we have

$$\frac{(x_1 + lr)^2}{a^2} + \frac{(y_1 + mr)^2}{b^2} = 1$$

or,
$$r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right) + 2r\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2}\right) + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = 0.$$

If the straight line touches the ellipse, then the roots of this quadratic equation in r will be equal.

Hence
$$\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2}\right)^2 = \left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right).$$

Eliminating l, m from the equations of the straight line and the above equation, we have

$$\left\{\frac{x_1(x - x_1)}{a^2} + \frac{y_1(y - y_1)}{b^2}\right\}^2$$
$$= \left\{\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{b^2}\right\}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right),$$

which may be written as $SS_1 = T^2$,

where $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $\quad S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$

and $\quad T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

This, being satisfied by any point on the tangent drawn through the point (x_1, y_1), gives the equation of the pair of tangents from the point (x_1, y_1) to the ellipse.

Cor. Proceeding in the same way, we get the equation of the pair of tangents from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as

$$SS_1 = T^2,$$

where $\quad S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $\quad S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $\quad T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

5.15. Chord of contact of tangents.

The *chord of contact* of tangents from an external point is the straight line joining the points of contact of the pair of tangents from the external point to the conic.

5.16. Equation of the chord of contact of tangents from an external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$.

Let $A(x_2, y_2)$ and $B(x_3, y_3)$ be the points of contact of the tangents from an external point $P(x_1, y_1)$ to the circle

$$x^2 + y^2 = a^2.$$

The straight line AB is the chord of contact.

Now the equation of the tangent PA at $A(x_2, y_2)$ is

$$xx_2 + yy_2 = a^2.$$

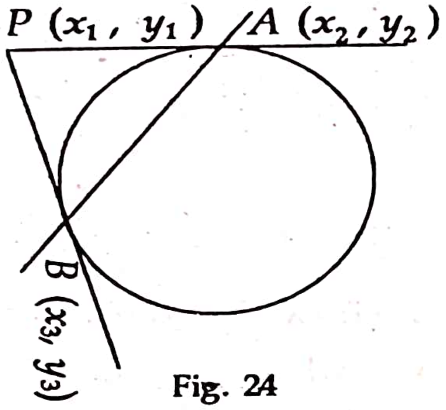

Fig. 24

Since it passes through the point $P(x_1, y_1)$, we have

$$x_1x_2 + y_1y_2 = a^2. \quad \ldots (1)$$

Similarly, since the tangent PB at $B(x_3, y_3)$ passes through the point $P(x_1, y_1)$, we have

$$x_1x_3 + y_1y_3 = a^2. \quad \ldots (2)$$

The relations (1) and (2) show that both (x_2, y_2) and (x_3, y_3) satisfy the equation

$$xx_1 + yy_1 = a^2;$$

which represents a straight line.

This is, therefore, the equation of the chord of contact.

Cor. The equation of the chord of contact of tangents from a point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

5.17. Equation of the chord of contact of tangents from an external point (x_1, y_1) to the parabola $y^2 = 4ax$.

Let $A(x_2, y_2)$ and $B(x_3, y_3)$ be the points of contact of the tangents from an external point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$.

The straight line AB is the chord of contact.

Now the equation of the tangent PA at $A(x_2, y_2)$ is

$$yy_2 = 2a(x + x_2).$$

Since it passes through the point $P(x_1, y_1)$, we have

$$y_1y_2 = 2a(x_1 + x_2). \qquad \ldots (1)$$

Similarly, since the tangent PB at $B(x_3, y_3)$ passes through the point $P(x_1, y_1)$, we have

$$y_1y_3 = 2a(x_1 + x_3). \qquad \ldots (2)$$

The relations (1) and (2) show that both (x_2, y_2) and (x_3, y_3) lie on the straight line $yy_1 = 2a(x + x_1)$.

This is the equation of the chord of contact AB.

5.18. Equation of the chord of contact of tangents from an external point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $A(x_2, y_2)$ and $B(x_3, y_3)$ be the points of contact of the tangents from an external point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Now the equation of the tangent PA at $A(x_2, y_2)$ is

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1.$$

Since it passes through the point $P(x_1, y_1)$, we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1. \quad \ldots (1)$$

Similarly, since the tangent PB at $B(x_3, y_3)$ passes through the point P, we have

$$\frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} = 1. \quad \ldots (2)$$

The relations (1) and (2) show that the two points A and B lie on the straight line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

This is the equation of the chord of contact AB.

Cor. The equation of the chord of contact of tangents from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

5.19. Illustrative Examples.

Ex. 1. *Find the angle between the two tangents from an external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$.*

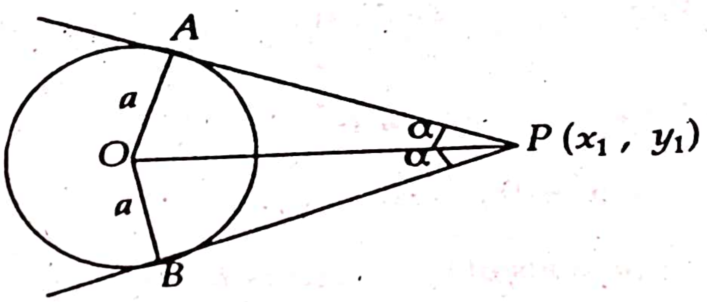

Fig. 25

Let the pair of tangents PA and PB from the external point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ be at an angle 2α.

Then $\angle APO = \angle BPO = \alpha$.

Since the tangent is perpendicular to the radius of the circle at the point of contact, therefore

$$\sin\alpha = \frac{OA}{OP} = \frac{a}{\sqrt{x_1^2 + y_1^2}}.$$

Hence the required angle between the pair of tangents is

$$2\alpha = 2\sin^{-1}\frac{a}{\sqrt{x_1^2 + y_1^2}}.$$

Ex. 2. *Prove that if the straight line $\lambda x + \mu y + \nu = 0$ touches the parabola $y^2 - 4px + 4pq = 0$, then $\lambda^2 q + \lambda\nu - p\mu^2 = 0$.*

Let the straight line $\lambda x + \mu y + \nu = 0$ ... (1)

touch the parabola $y^2 - 4px + 4pq = 0$ ... (2)

at the point (x_1, y_1).

The equation of the tangent to the parabola at (x_1, y_1) is

$$yy_1 - 2p(x + x_1) + 4pq = 0$$

or, $2px - yy_1 + 2p(x_1 - 2q) = 0.$ ... (3)

Therefore the equations (1) and (3) are identical.

Hence $\frac{2p}{\lambda} = -\frac{y_1}{\mu} = \frac{2p(x_1 - 2q)}{\nu}$

or, $x_1 = 2q + \frac{\nu}{\lambda}$ and $y_1 = -\frac{2p\mu}{\lambda}.$

Now the point (x_1, y_1) lies on the parabola (2).

Therefore $y_1^2 - 4px_1 + 4pq = 0$

or, $\frac{4p^2\mu^2}{\lambda^2} - 8pq - \frac{4p\nu}{\lambda} + 4pq = 0$

or, $-4p(\lambda^2 q + \lambda\nu - p\mu^2) = 0$

or, $\lambda^2 q + \lambda\nu - p\mu^2 = 0$, since $p \neq 0$.

Ex. 3. *Find the equations of the tangents to the conic*

$$x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$$

which are parallel to the straight line $x + 4y = 0$. [B. H. 1983, 1986]

Any straight line parallel to the straight line $x + 4y = 0$ is

$x + 4y + k = 0$, where k is a constant

or, $x = -(4y + k).$ ... (

Putting (1) in the equation of the conic, we get

$$(4y+k)^2-4y(4y+k)+3y^2+5(4y+k)-6y+3=0$$

or, $3y^2+2y(2k+7)+(k^2+5k+3)=0.$

If the straight line (1) touches the conic, the roots of the above equation will be equal.

Therefore $4(2k+7)^2=4.3(k^2+5k+3)$

or, $k^2+13k+40=0$

or, $(k+5)(k+8)=0$, whence $k=-5,-8$.

Hence the required tangents are

$$x+4y-5=0 \text{ and } x+4y-8=0.$$

Ex. 4. *Find the equation to the common tangent of the circle $x^2+y^2=4ax$ and the parabola $y^2=4ax$.*

Let $y=mx+c$ be the common tangent.

Since it touches the parabola, we have

$$c=a/m. \qquad \ldots \qquad (1)$$

Since it touches the circle, its distance from the centre $(2a, 0)$ is equal to the radius. In other words, we have

$$\frac{2am+c}{\sqrt{1+m^2}}=\pm 2a$$

or, $(2am+a/m)^2=4a^2(1+m^2)$, by (1)

or, $4a^2m^2+\frac{a^2}{m^2}+4a^2=4a^2+4a^2m^2$

or, $\frac{a^2}{m^2}=0$, whence $\frac{1}{m}=0$, since $a\neq 0$.

Hence the equation of the common tangent is $x=\frac{y}{m}-\frac{c}{m}=0.$

Ex. 5. *Prove that the length of the tangent drawn from any point on the circle $x^2+y^2+2gx+2fy+c=0$ to the circle $x^2+y^2+2gx+2fy+c'=0$ is $(c'-c)^{1/2}$.*

The length of the tangent from any point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c'=0$ is $(x_1{}^2+y_1{}^2+2gx_1+2fy_1+c')^{1/2}$.

Now the point (x_1, y_1) lies on the circle

$$x^2+y^2+2gx+2fy+c=0.$$

Therefore we have $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$

Thus $\quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c.$

Hence the length of the tangent is $(c' - c)^{1/2}$.

Ex. 6. (*a*) *Find the point of intersection of the tangents at the points* $(at_1^2, 2at_1)$ *and* $(at_2^2, 2at_2)$ *to the parabola* $y^2 = 4ax$.

(*b*) *Show that the locus of the point of intersection of tangents at two points on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *where the difference of their eccentric angles is* 2α, *is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\alpha.$$

(*a*) Let (x_1, y_1) be the point of intersection of the two tangents. The equation of the chord of contact of the tangents from (x_1, y_1) is

$$yy_1 = 2a(x + x_1).$$

Since it passes through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, we have

$$2at_1y_1 = 2a(at_1^2 + x_1)$$

or, $\quad t_1y_1 = x_1 + at_1^2 \quad \ldots (1)$

and $\quad t_2y_1 = x_1 + at_2^2. \quad \ldots (2)$

Solving (1) and (2), we have $x_1 = at_1t_2$ and $y_1 = a(t_1 + t_2)$.

Hence the point of intersection is $\{at_1t_2, a(t_1 + t_2)\}$.

(*b*) Let (x_1, y_1) be the point of intersection of the two tangents at the two points whose eccentric angles are ϕ and ϕ'.

Then $\phi - \phi' = 2\alpha$.

The equation of the tangent at (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots (1)$$

is $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \quad \ldots (2)$$

The equation of the chord of the ellipse (1) joining the two points ϕ and ϕ' is

$$\frac{x}{a}\cos\frac{\phi + \phi'}{2} + \frac{y}{b}\sin\frac{\phi + \phi'}{2} = \cos\frac{\phi - \phi'}{2} = \cos\alpha. \quad \ldots (3)$$

According to the problem, the equations (2) and (3) are identical.

Therefore $\dfrac{x_1/a^2}{\frac{1}{a}\cos\frac{\phi+\phi'}{2}} = \dfrac{y_1/b^2}{\frac{1}{b}\sin\frac{\phi+\phi'}{2}} = \dfrac{1}{\cos\alpha}$

or, $x_1 = \dfrac{a\cos\frac{1}{2}(\phi+\phi')}{\cos\alpha}$ and $y_1 = \dfrac{b\sin\frac{1}{2}(\phi+\phi')}{\cos\alpha}$...

Hence $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = \dfrac{1}{\cos^2\alpha} = \sec^2\alpha$.

Thus the locus of (x_1, y_1) is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \sec^2\alpha$.

Examples V(A)

1. Find the equation of the tangent

(*i*) at the point (1, 3) to the parabola $y^2 = 9x$;

(*ii*) at the point (1, 4) to the ellipse $3x^2 + 7y^2 = 115$.

2. Two circles both touch the axis of y and intersect at the points (1, 0) and (2, –1). Show that they both touch the line $y+2=0$.

3. Find the common tangents of the circles $x^2 + y^2 + 4x + 2y - 4 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$.

4. Find the equation of the common tangent to the two parabolas

(*i*) $y^2 = 32x$ and $x^2 = 108y$. (*ii*) $y^2 = 4ax$ and $x^2 = 4by$.

5. Find the equations of the straight lines which touch the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$.

6. Find the equation of the common tangent to the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{b^2} - \frac{y^2}{a^2} = -1.$$

Also find the length of the common tangent.

7. (*a*) A tangent to the parabola $y^2 + 4bx = 0$ meets the parabola $y^2 = 4ax$ at P and Q. Prove that the locus of the mid-point of PQ is

$$y^2(2a+b) = 4a^2x.$$

(*b*) A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = c^2$ in two points P and Q. Prove that the middle point of PQ lies on a parabola. [*N. B. H. 2005*]

8. Find the equations of the tangents to the circle $x^2 + y^2 + 8x + 10y - 4 = 0$ which are parallel to the straight line

$$x + 2y + 3 = 0.$$

9. (*a*) Obtain the equations of the tangents to the circle $x^2 + y^2 = 25$ which pass through the point (13, 0).

(*b*) Show that the equation of the pair of tangents from the point (–2, 1) to the parabola $y^2 = 8x$ is $2x^2 - 2y^2 - xy + 9x + 2y + 8 = 0$

10. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

11. (*a*) Find the equation of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ which makes an angle of 60° with the *x*-axis.

(*b*) Find the equation of the straight lines which passes through the point (18, 12) and which touches the ellipse $x^2 + 8y^2 = 32$.

12. Show that the straight line $x + y = 2$ touches the circle $x^2 + y^2 = 2$ and $x^2 + y^2 + 3x + 3y = 8$ at the same point.

13. Prove that the straight line $lx + my + n = 0$ touches

(*i*) the circle $x^2 + y^2 = a^2$, if $n^2 = a^2(l^2 + m^2)$;

(*ii*) the parabola $y^2 = 4ax$, if $ln = am^2$;

(*iii*) the ellipse $x^2/a^2 + y^2/b^2 = 1$, if $n^2 = a^2l^2 + b^2m^2$;

(*iv*) the hyperbola $x^2/a^2 - y^2/b^2 = 1$, if $n^2 = a^2l^2 - b^2m^2$.

14. (*a*) Show that the straight line $y = x + \sqrt{\frac{5}{6}}$ touches the ellipse

$$2x^2 + 3y^2 = 1.$$

(*b*) If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes intercepts of lengths *h* and *k* on the axes, then prove that $a^2/h^2 + b^2/k^2 = 1$.

(*c*) Show that the straight line $x \cos \alpha + y \sin \alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$.

15. Show that the equation of the circle which touches the co-ordinate axes in the first quadrant and whose centre lies on the straight line $lx + my + n = 0$ is $(l + m)^2 (x^2 + y^2) + 2n(l + m)(x + y) + n^2 = 0$.

16. The length of the tangent from (f, g) to the circle $x^2 + y^2 = 6$ is twice the length of the tangent to the circle $x^2 + y^2 + 3x + 3y = 0$.

Show that $$f^2 + g^2 + 4f + 4g + 2 = 0.$$

17. (a) Prove that the two parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect at an angle of $\tan^{-1}\dfrac{3\sqrt[3]{ab}}{2\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)}$ at one of their common points.

(b) Show that the angle between the tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the circle $x^2+y^2=ab$ is $\tan^{-1}\dfrac{a-b}{\sqrt{ab}}$ at their point of intersection.

18. A circle cuts the parabola $y^2 = 4ax$ at right angles and passes through the focus. Show that its centre lies on the curve

$$y^2(a+2x) = a(a+3x)^2.$$

[N. B. H. 2003]

[Let $C(\alpha, \beta)$ be the centre of the circle. Then its equation is $(x-\alpha)^2+(y-\beta)^2=(a-\alpha)^2+\beta^2$. Let the circle meet the parabola at $P(at^2, 2at)$. Here the tangent to the parabola at P passes through C. Also P lies on the circle.]

19. Two straight lines are at right angles to one another, one of them touches the parabola $y^2 = 4a(x+a)$ and the other touches the parabola $y^2 = 4a'(x+a')$. Show that the point of intersection of the straight lines will lie on the straight line $x+a+a'=0$.

20. Find the locus of the point of intersection of the tangents of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the points θ and $\left(\dfrac{\pi}{2}+\theta\right)$.

21. (a) Prove that the orthocentre of a triangle formed by three tangents to a parabola lies on the directrix.

(b) Prove that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

[B. H. 1994]

22. The product of the lengths of the tangents drawn from a point P to the parabola $y^2 = 4ax$ is equal to the product of the focal distance of P and the latus rectum. Prove that the locus of P is the parabola

$$y^2 = 4a(x+a).$$

23. (a) Show that the locus of the point of intersection of a pair of perpendicular tangents to an ellipse is a circle.

(b) An ellipse slides between two straight lines at right angles to each other. Show that the locus of its centre is a circle.

(c) Show that the locus of the point of intersection of a pair of perpendicular tangents to the parabola $x^2 = 4by$ is the directrix.

(d) Find the locus of the point of intersection of two tangents to the parabola $y^2 = 4ax$ such that their chord of contact subtends a right angle at the vertex.

[V.H. 1997]

(c) Prove that the locus of a point from which perpendicular tangents can be drawn to the circle $x^2+y^2=a^2$ is the circle

$$x^2+y^2=2a^2.$$

24. (a) Two tangents drawn to the parabola $y^2=4ax$ meet at an angle of 45°. Show that the locus of their point of intersection is

$$(x+a)^2=y^2-4ax.$$ [C. H. 2002 ; B. H. 2006]

(b) If α be the angle between a pair of tangents to the parabola $y^2=4ax$, then show that the locus of their point of intersection is

$$(x+a)^2\tan^2\alpha=y^2-4ax.$$ [C. H. 2012]

25. If α be the angle between a pair of tangents to the ellipse $x^2/a^2+y^2/b^2=1$, then show that the locus of their point of intersection is $(x^2+y^2-a^2-b^2)^2\tan^2\alpha=4(b^2x^2+a^2y^2-a^2b^2)$.

26. Show that the foot of the perpendicular from a focus to any tangent to the hyperbola $x^2/a^2-y^2/b^2=1$ lies on the circle

$$x^2+y^2=a^2.$$ [B. H. 2002 ; C. H. 2004]

27. (a) Tangents are drawn to the parabola $y^2=4ax$ at the points whose abscissae are in the ratio $p:1$. Show that the locus of their point of intersection is a parabola. [C. H. 1998]

(b) Show that the locus of the point of intersection of tangents to the parabola $y^2=4ax$ at points whose ordinates are in the ratio $p^2:q^2$ is $$y^2=\left(\frac{p^2}{q^2}+\frac{q^2}{p^2}+2\right)ax.$$ [C. H. 1974]

28. Find the equation of the chord of contact of the point (6, 5) with respect to the ellipse $4x^2+9y^2=36$.

29. Tangents are drawn from any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the circle $x^2+y^2=r^2$. Prove that the chords of contact are tangents to the ellipse $a^2x^2+b^2y^2=r^4$. [B. H. 1990]

30. (a) Tangents are drawn from the point (x_1, y_1) to the circle $x^2+y^2=a^2$; prove that the area of the triangle formed by them and the straight line joining their points of contact is $\dfrac{a(x_1^2+y_1^2-a^2)^{3/2}}{x_1^2+y_1^2}$.

[In Fig. 25, area of ΔPAB is $\frac{1}{2}AP.AP\sin 2\alpha=AP^2\sin\alpha\cos\alpha$.]

(b) Find the area of the triangle formed by the tangents from the point (h, k) to the parabola $y^2=4ax$ and the chord of contact.

31. An ellipse is rotated through a right angle in its own plane about its centre, which is fixed. Prove that the locus of the point of intersection of a tangent to the ellipse in its original position with the tangent at the same point of the curve in the new position is

$$(x^2+y^2)(x^2+y^2-a^2-b^2)=2(a^2-b^2)xy.$$ [N. B. H. 2000]

32. The tangent at any point θ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle at two points which subtend a right angle at the centre. Show that the eccentricity of the ellipse is given by $e^{-2}=1+\sin^2\theta$.

[The tangent $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ meets the circle $x^2+y^2=a^2$ at two points which subtend a right angle at the origin. Hence the straight lines represented by $x^2+y^2=a^2\left(\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}\right)^2$ are perpendicular.]

Answers

1. (i) $3x-2y+3=0$. (ii) $3x+28y-115=0$. 3. $y=2,\ 3y=4x-10$.

4. (i) $2x+3y+36=0$. (ii) $a^{1/3}x+b^{1/3}y+a^{2/3}b^{2/3}=0$.

5. $y=2x+1,\ y+2x+1=0$. 6. $x\pm y=\pm\sqrt{a^2-b^2}$; $\sqrt{2}\frac{a^2+b^2}{\sqrt{a^2-b^2}}$.

8. $x+2y-1=0,\ x+2y+29=0$. 9. (a) $12y=\pm 5(x-13)$.

10. $2x+y+1=0,\ (\frac{1}{2},-2)$; $x-2y+8=0,\ (8,8)$.

11. (a) $y=\sqrt{3}\,x\pm\sqrt{3a^2+b^2}$. (b) $x-y=6,\ 35x-73y+246=0$.

20. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$ 23. (d) $x+4a=0$. 28. $8x+15y=12$

30. (b) $\frac{1}{2a}(k^2-4ah)^{3/2}$ square units.

5.20. Normals.

The *normal* to a curve at a point is the straight line which passes through the point and which is perpendicular to the tangent to the curve at that point.

5.21. Equation of the normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point (x_1, y_1).

The normal to the circle at the point (x_1, y_1) is the straight line through the point perpendicular to the tangent to the circle at the point (x_1, y_1).

The tangent at the point (x_1, y_1) to the given circle is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

Its slope is $\left(-\frac{x_1+g}{y_1+f}\right)$.

Then the slope of the normal will be $\frac{y_1+f}{x_1+g}$.

Hence the equation of the normal at the point (x_1, y_1) is

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1).$$

32. The tangent at any point θ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle at two points which subtend a right angle at the centre. Show that the eccentricity of the ellipse is given by $e^{-2}=1+\sin^2\theta$.

[The tangent $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ meets the circle $x^2+y^2=a^2$ at two points which subtend a right angle at the origin. Hence the straight lines represented by $x^2+y^2=a^2\left(\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}\right)^2$ are perpendicular.]

Answers

1. (i) $3x-2y+3=0$. (ii) $3x+28y-115=0$. 3. $y=2,\ 3y=4x-10$.
4. (i) $2x+3y+36=0$. (ii) $a^{1/3}x+b^{1/3}y+a^{2/3}b^{2/3}=0$.
5. $y=2x+1,\ y+2x+1=0$. 6. $x\pm y=\pm\sqrt{a^2-b^2}$; $\sqrt{2}\frac{a^2+b^2}{\sqrt{a^2-b^2}}$.
8. $x+2y-1=0,\ x+2y+29=0$. 9. (a) $12y=\pm 5(x-13)$.
10. $2x+y+1=0,\ (\frac{1}{2},-2)$; $x-2y+8=0,\ (8,8)$.
11. (a) $y=\sqrt{3}\,x\pm\sqrt{3a^2+b^2}$. (b) $x-y=6,\ 35x-73y+246=0$.
20. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$. 23. (d) $x+4a=0$. 28. $8x+15y=12$.
30. (b) $\frac{1}{2a}(k^2-4ah)^{3/2}$ square units.

5.20. Normals.

The *normal* to a curve at a point is the straight line which passes through the point and which is perpendicular to the tangent to the curve at that point.

5.21. Equation of the normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point (x_1, y_1).

The normal to the circle at the point (x_1, y_1) is the straight line through the point perpendicular to the tangent to the circle at the point (x_1, y_1).

The tangent at the point (x_1, y_1) to the given circle is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

Its slope is $\left(-\frac{x_1+g}{y_1+f}\right)$.

Then the slope of the normal will be $\frac{y_1+f}{x_1+g}$.

Hence the equation of the normal at the point (x_1, y_1) is

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1).$$

Second method.

The tangent to the circle at the point (x_1, y_1) is perpendicular to the radius of the circle through (x_1, y_1). So this radius is the normal to the circle (x_1, y). Hence the normal to the circle at (x_1, y_1) passes through the centre $(-g, -f)$ of the circle and the point (x_1, y_1). Therefore its equation is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1).$$

Cor. The equation of the normal to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) is $xy_1 - yx_1 = 0$.

5.22. Equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is $yy_1 = 2a(x + x_1)$ whose slope is $\frac{2a}{y_1}$.

Then the slope of the normal will be $\left(\frac{-y_1}{2a}\right)$.

Hence the equation of the normal at the point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

Cor. 1. The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y - 2at = -t(x - at^2)$$

or, $$tx + y = 2at + at^3.$$

Cor. 2. If m be the slope of the normal at the point (x_1, y_1), that is, if $m = -\frac{y_1}{2a}$, then the equation of the normal may be written as

$$y + 2am = m(x - am^2); \text{ since } x_1 = \frac{y_1^2}{4a} = \frac{4a^2m^2}{4a} = am^2$$

or, $y = mx - 2am - am^3$.

5.23. Co-normal points of a parabola.

Let (x_1, y_1) be a given point on the plane of the parabola

$$y^2 = 4ax.$$

The equation of the normal at $(at^2, 2at)$ to $y^2 = 4ax$ is

$$y + tx = 2at + at^3. \qquad \ldots (1)$$

If it passes through the point (x_1 , y_1), then

$$y_1 + tx_1 = 2at + at^3$$

or, $at^3 + t (2a - x_1) - y_1 = 0$ . ... (2)

This equation is a cubic in t and hence this gives three values of t. Thus *three normals can be drawn from a given point to a parabola*. In other words, *there are three points on the parabola, the normals at which pass through the given point*. These points are called *co-normal points*.

If the roots of the equation (2) be t_1, t_2, t_3, then the equations of the three normals are obtained from (1) by putting these values of t.

Again, we see, from (2), that the sum of the roots is zero,

that is, $$t_1 + t_2 + t_3 = 0.$$

The feet of the three normals are $(at_1^2 , 2at_1)$, $(at_2^2 , 2at_2)$, $(at_3^2 , 2at_3)$. The sum of the ordinates of the co-normal points is

$$2at_1 + 2at_2 + 2at_3 = 2a (t_1 + t_2 + t_3) = 0.$$

Hence *the sum of the ordinates of the feet of the normals passing through the given point is zero.*

5.24. Equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1 , y_1).

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1 , y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 .$$

The slope of the tangent is $\left(- \frac{b^2 x_1}{a^2 y_1} \right)$.

Therefore the slope of the normal will be $\frac{a^2 y_1}{b^2 x_1}$.

Then the equation of the normal at the point (x_1 , y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1) .$$

This can be put as

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}$$

or, $$\frac{a^2}{x_1}x - \frac{b^2}{y_1}y = a^2 - b^2.$$

Cor. The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.

5.25. Equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

Proceeding exactly in the same way as in the case of ellipse, we get the equation of the normal to the hyperbola at the point (x_1, y_1) as $$\frac{a^2}{x_1}x + \frac{b^2}{y_1}y = a^2 + b^2.$$

The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\phi, b\tan\phi)$ is $\frac{ax}{\sec\phi} + \frac{by}{\tan\phi} = a^2 + b^2$.

5.26. Co-normal points of an ellipse.

Let us consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equation of the normal at the point θ is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2. \quad \dots (1)$$

If it passes through the point $P(x_1, y_1)$, then we have

$$ax_1\sec\theta - by_1\operatorname{cosec}\theta = a^2 - b^2. \quad \dots (2)$$

Putting $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$, where $t = \tan\frac{\theta}{2}$, we get

$$ax_1\frac{1+t^2}{1-t^2} - by_1\frac{1+t^2}{2t} = a^2 - b^2$$

or, $2ax_1(1+t^2)t - by_1(1+t^2)(1-t^2) = 2(a^2 - b^2)t(1-t^2)$.

This is a biquadratic equation in t and hence this will give four values for t, that is, for $\tan\frac{1}{2}\theta$.

Hence there are four values of θ satisfying (2). Substituting these values of θ in (1), we get the four normals from the point $P(x_1, y_1)$ to the ellipse.

Arranging the equation in t, we get

$$by_1t^4 + 2(ax_1 + a^2 - b^2)t^3 + 2(ax_1 - a^2 + b^2)t - by_1 = 0.$$

If the roots of this equation be t_1, t_2, t_3 and t_4, then we have

$$\Sigma t_1 = \frac{-2(ax_1 + a^2 - b^2)}{by_1}; \Sigma t_1 t_2 = 0;$$

$$\Sigma t_1 t_2 t_3 = \frac{-2(ax_1 - a^2 + b^2)}{by_1}; t_1 t_2 t_3 t_4 = -1.$$

In Trigonometry we have the relation

$$\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - \Sigma t_1 t_2 + t_1 t_2 t_3 t_4}$$

$$= \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - 1}.$$

Therefore $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ is an odd multiple of $\frac{1}{2}\pi$.

Hence $(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ is an odd multiple of π.

Thus *four normals can be drawn from a point to an ellipse and the sum of the eccentric angles of the feet of the four normals is an odd multiple of* π

5.27. Curve through the co-normal points of an ellipse.

The equation of the normal at the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2}{x_1}x - \frac{b^2}{y_1}y = a^2 - b^2$.

If this normal passes through the point (h, k), then we have

$$\frac{a^2h}{x_1} - \frac{b^2k}{y_1} = a^2 - b^2.$$

Therefore the foot of the normal lies on the curve

$$\frac{a^2h}{x} - \frac{b^2k}{y} = a^2 - b^2$$

or, $\quad (a^2 - b^2)\, xy + b^2 kx - a^2hy = 0.$

This is the equation of a curve and the feet of the four normals from the point (h, k) lie on this curve.

5.28. Four normals from a point to a hyperbola.

Let the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Its normal at a point θ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2. \qquad \ldots (1)$$

If it passes through the point (x_1, y_1), then we have

$$\frac{ax_1}{\sec\theta} + \frac{by_1}{\tan\theta} = a^2 + b^2. \qquad \ldots (2)$$

Let us put $t = \tan\frac{1}{2}\theta$, so that

$$\cos\theta = \frac{1 - t^2}{1 + t^2} \text{ and } \tan\theta = \frac{2t}{1 - t^2}.$$

Then (2) may be written as

$$ax_1 \frac{1 - t^2}{1 + t^2} + by_1 \frac{1 - t^2}{2t} = a^2 + b^2$$

or, $2ax_1(1 - t^2)\, t + by_1(1 - t^4) = 2t(a^2 + b^2)(1 + t^2).$

This being a fourth degree equation in t, we shall have four values of θ corresponding to these four values of t. These four values of θ when substituted in (1) give four normals drawn from the point (x_1, y_1) to the hyperbola.

5.29. Sub-tangent and sub-normal of a parabola.

Let P be a point on the parabola $y^2 = 4ax$. Let the tangent PT and the normal PG meet the axis at the points T and G respectively.

Let PN be the ordinate of P. Then TN, the projection of the tangent on the axis of the parabola is called the *sub-tangent* and NG, the projection of the normal on the axis is called the *sub-normal* of the point P.

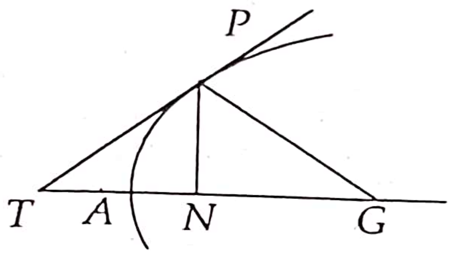

Fig. 26

Let the co-ordinates of P be $(at^2, 2at)$; then the equation of the normal at P is

$$y + tx = 2at + at^3.$$

It meets the axis, $y = 0$, of the parabola at $x = 2a + at^2$.

Thus the co-ordinates of G are $(2a + at^2, 0)$.

Therefore the sub-normal is

$$NG = AG - AN = 2a + at^2 - at^2 = 2a,$$ which is constant.

Again, the tangent at $P(at^2, 2at)$ is

$$ty = x + at^2.$$

It meets the axis, $y = 0$, at $x = -at^2$.

Therefore $AT = AN = at^2$.

Thus *the sub-normal is of constant length and the sub-tangent is bisected at the vertex of a parabola.*

5.30. Illustrative Examples.

Ex. 1. *If the normal at the point $(at_1{}^2, 2at_1)$ on the parabola $y^2 = 4ax$ meets it again at the point $(at^2, 2at)$, then show that $t = -t_1 - 2/t_1$.*

The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at_1{}^2, 2at_1)$ is $y - 2at_1 = -t_1(x - at_1{}^2)$.

Since it passes through the point $(at^2, 2at)$, we have

$$2at - 2at_1 = -t_1(at^2 - at_1{}^2)$$

or, $\quad 2a(t - t_1) = -at_1(t^2 - t_1^2)$

or, $\quad 2 = -t_1(t + t_1),$ $\qquad$ since $t \neq t_1$

or, $\quad t + t_1 = -2/t_1$, whence $t = -t_1 - 2/t_1$.

Ex. 2. *Show that the straight line $lx + my = n$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if* $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$.

Let the given straight line $\quad lx + my = n \qquad \ldots$ (1)
be a normal to the ellipse at the point (x_1, y_1).

The equation of the normal to the ellipse at the point (x_1, y_1) is

$$\frac{a^2}{x_1}x - \frac{b^2}{y_1}y = a^2 - b^2. \qquad \ldots (2)$$

The equations (1) and (2) are identical.

Therefore $\quad \frac{l}{a^2/x_1} = \frac{m}{-b^2/y_1} = \frac{n}{a^2 - b^2}$

or, $x_1 = \frac{a^2 n}{l(a^2 - b^2)}$ and $y_1 = -\frac{b^2 n}{m(a^2 - b^2)}$.

Now, (x_1, y_1) lies on the ellipse.

Therefore $\quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

or, $\quad \frac{a^2 n^2}{l^2(a^2 - b^2)^2} + \frac{b^2 n^2}{m^2(a^2 - b^2)^2} = 1$

or, $\quad \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$.

Ex. 3. *Find the condition that the normals at the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) on the ellipse $x^2/a^2 + y^2/b^2 = 1$ will be concurrent.*

Let the three normals meet at the point (α, β). The equation of the normal to the ellipse at (x_1, y_1) is

$$\frac{a^2}{x_1}x - \frac{b^2}{y_1}y = a^2 - b^2.$$

Since it passes through the point (α, β), we have

$$\frac{a^2}{x_1}\alpha - \frac{b^2}{y_1}\beta = a^2 - b^2.$$

or, $\quad a^2 y_1 \alpha - b^2 x_1 \beta = (a^2 - b^2)x_1 y_1. \qquad \ldots$ (1)

Similarly, for the points (x_2, y_2) and (x_3, y_3), we have respectively

$$a^2 y_2 \alpha - b^2 x_2 \beta = (a^2 - b^2)\, x_2 y_2 \qquad \ldots (2)$$

and $$a^2 y_3 \alpha - b^2 x_3 \beta = (a^2 - b^2)\, x_3 y_3 . \qquad \ldots (3)$$

Eliminating α and β from (1), (2) and (3), we get the required condition as

$$\begin{vmatrix} a^2 y_1 & -b^2 x_1 & (a^2 - b^2)\, x_1 y_1 \\ a^2 y_2 & -b^2 x_2 & (a^2 - b^2)\, x_2 y_2 \\ a^2 y_3 & -b^2 x_3 & (a^2 - b^2)\, x_3 y_3 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0 .$$

Ex. 4. *If the normal to the hyperbola $xy = c^2$ at the point $\left(ct_1, \frac{c}{t_1}\right)$ meets the curve again at the point $\left(ct_2, \frac{c}{t_2}\right)$, then show that $t_1^3 t_2 + 1 = 0$.*

The tangent to the hyperbola $xy = c^2$ at the point $\left(ct_1, \frac{c}{t_1}\right)$ is $\frac{1}{2}\left(x\frac{c}{t_1} + yct_1\right) = c^2$ whose slope is $\left(-\frac{1}{t_1^2}\right)$.

Therefore the equation of the normal at the point $\left(ct_1, \frac{c}{t_1}\right)$ is

$$y - \frac{c}{t_1} = t_1^2 (x - ct_1) .$$

If it passes through the point $(ct_2, c/t_2)$, then we have

$$\frac{c}{t_2} - \frac{c}{t_1} = t_1^2 (ct_2 - ct_1)$$

or, $$\frac{c(t_1 - t_2)}{t_1 t_2} + ct_1^2 (t_1 - t_2) = 0$$

or, $$t_1^2 + \frac{1}{t_1 t_2} = 0, \quad \text{since } t_1 \neq t_2$$

or, $$t_1^3 t_2 + 1 = 0 .$$

Ex. 5. *If the normal to an ellipse at the point P meets the major and the minor axes at G and g respectively, then prove that PG . Pg = SP . S'P, S and S' being the foci of the ellipse.* [C. H. 1960]

The equation of the normal at the point $P(a\cos\phi,\ b\sin\phi)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $ax\sec\phi - by\operatorname{cosec}\phi = a^2-b^2$.

It meets the major axis, $y=0$, at $G\left(\frac{a^2-b^2}{a}\cos\phi,\ 0\right)$

and the minor axis, $x=0$, at $g\left(0,\ \frac{b^2-a^2}{b}\sin\phi\right)$.

$$\text{Therefore } PG^2 . Pg^2 = \left(\frac{b^4}{a^2}\cos^2\phi + b^2\sin^2\phi\right)\left(a^2\cos^2\phi + \frac{a^4}{b^2}\sin^2\phi\right)$$

$$= \frac{b^2}{a^2}\left(b^2\cos^2\phi + a^2\sin^2\phi\right)\cdot\frac{a^2}{b^2}\left(b^2\cos^2\phi + a^2\sin^2\phi\right)$$

$$= (a^2\sin^2\phi + b^2\cos^2\phi)^2$$

or, $$PG . Pg = a^2\sin^2\phi + b^2\cos^2\phi$$

$$= a^2(1-\cos^2\phi) + a^2(1-e^2)\cos^2\phi$$

$$= a^2(1-e^2\cos^2\phi)$$

$$= a(1-e\cos\phi) . a(1+e\cos\phi) = SP . S'P .$$

Ex. 6. *The normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of the chords $lx+my=1$ and $l'x+m'y=1$ will be concurrent, if $a^2ll' = b^2mm' = -1$.*

Two points of the four concurrent normals lie on the straight line

$$lx + my = 1. \quad \ldots (1)$$

Let the other two lie on the straight line $l'x+m'y=1$. ... (2)

Hence the four feet lie on $(lx+my-1)(l'x+m'y-1)=0$. ... (3)

Since the four feet lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, ... (4)

equations (3) and (4) must be identical.

Comparing the terms in x^2 and y^2 and the constant terms, we get

$$\frac{ll'}{\frac{1}{a^2}} = \frac{mm'}{\frac{1}{b^2}} = \frac{1}{-1}.$$

Therefore $a^2ll' = b^2mm' = -1$.

Examples V (B)

1. Find the equation of the normal

(i) at the point (1, 3) to the parabola $y^2 = 9x$;

(ii) at the point (1, 4) to the ellipse $3x^2 + 7y^2 = 115$;

(iii) at the point (1, 1) to the hyperbola $x^2 - 2y^2 + x + y = 1$.

2. Prove that the normal to the circle $x^2 + y^2 - 5x + 2y = 48$ at the point (5, 6) is a tangent to the parabola $5y^2 + 448x = 0$.

3. Prove that the straight line $lx + my + n = 0$ is a normal to

(i) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$.

(ii) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$.

4. Prove that the straight line $4ax + 3by = 12c$ will be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $5c = a^2 e^2$, e being the eccentricity.

5. Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the normals at which pass through a given point $(h, 0)$ on the major axis. [*C. H. 1974*]

6. Prove that the straight line $2x + 4y = 9$ is a normal to the parabola $y^2 = 8x$. Also find the co-ordinates of the foot of the normal.

7. Show that the chord of the parabola $y^2 = 4ax$ whose equation is $y = x\sqrt{2} - 4a\sqrt{2}$ is a normal to the curve and that its length is $6a\sqrt{3}$.

8. Show that the normals at the points (8, 8) and (2, 4) of the parabola $y^2 = 8x$ intersect on the curve.

9. Find the point of the parabola $y^2 = 8x$ at which the normal is inclined at 60° to the axis of the parabola.

10. (a) If a normal chord of a parabola subtends a right angle at the vertex, then show that its inclination to the x-axis is $\tan^{-1}\sqrt{2}$.

(b) Show that the normal chord of a parabola at the point, whose ordinate is equal to its abscissa, subtends a right angle at the focus. [*B. H. 2003*]

11. Prove that the feet of the normals from the point (h, k) to the parabola $y^2 = 4ax$ lie on the curve $xy - (h - 2a)y = 2ak$.

12. Show that the circle passing through the feet of the normals from a point to a parabola cuts the parabola at the vertex.

13. (*a*) Prove that the locus of the point of intersection of the normals to the parabola $y^2 = 4ax$ at the extremities of the focal chord is the parabola $y^2 = a(x-3a)$ which is the locus of the point of intersection of two perpendicular normals to the parabola $y^2 = 4ax$.

(*b*) *TP* and *TQ* are tangents to the parabola $y^2 = 4ax$ drawn from a variable point *T*. If *TP* and *TQ* be always perpendicular to each other, then show that the locus of the point of intersection of the normals to the parabola at *P* and *Q* is the parabola $y^2 = a(x-3a)$.

14. If the normal to the ellipse $\frac{1}{14}x^2 + \frac{1}{5}y^2 = 1$ at the point θ cuts the curve again at the point 2θ, then prove that $\cos\theta = -\frac{2}{3}$.

15. In an ellipse the normal at an extremity of the latus rectum passes through an extremity of the minor axis. Prove that $e^4 + e^2 = 1$, where *e* is the eccentricity. [C. H. 1993]

16. Any ordinate *NP* of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in *Q*; prove that the locus of the intersection of the normals at *P* and *Q* is the circle $x^2 + y^2 = (a+b)^2$. [C. H. 1999]

17. Show that the normals at the ends of each of a series of parallel chords of a parabola intersect on a fixed straight line which is a normal to the parabola.

18. (*a*) If the tangents at two points of the parabola $y^2 = 4ax$ intersect at the point (x_1, y_1), then show that the corresponding normals will intersect at the point $\left(2a - x_1 + \frac{y_1^2}{a}, -\frac{x_1 y_1}{a}\right)$.

(*b*) The tangents at the extremities of a normal chord of the parabola $y^2 = 4ax$ meet in a point *T*. Show that the locus of *T* is the curve

$$(x + 2a)y^2 + 4a^3 = 0.$$ [B. H. 1995]

19. Show that the normal $y = mx - 2am - am^3$ of the parabola $y^2 = 4ax$ intersects the parabola again at an angle $\tan^{-1}\left(\frac{1}{2}m\right)$.

20. If the normals at two points *A* and *B* of a parabola intersect on the curve, then show that the straight line *AB* passes through a fixed point. [C. H. 1967 ; B. H. 1991]

21. The normals at the ends of the latus rectum of the parabola $y^2 = 4ax$ meet the curve again in Q and Q′. Prove that $QQ' = 12a$.

[C. H. 1992]

22. Show that the locus of points such that two of the three normals drawn from them to the parabola $y^2 = 4ax$ coincide is

$$27ay^2 = 4(x - 2a)^3.$$ [N. B. H. 2007]

23. If three normals from a point to the parabola $y^2 = 4ax$ cut the axis in points whose distances from the vertex are in A. P., then show that the point lies on the curve $27ay^2 = 2(x - 2a)^3$. [N. B. H. 1994]

[Let the feet of the normals from the point (h, k) be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$. Then t_1, t_2, t_3 are the roots of the equation

$$at^3 + (2a - h)t - k = 0.$$

Therefore $\Sigma t_1 = 0$, $\Sigma t_1 t_2 = (2a - h)/a$, $t_1 t_2 t_3 = k/a$.

The equation of the normal to the parabola at $(at_1^2, 2at_1)$ is

$$y + t_1 x = 2at_1 + at_1^3.$$

Putting $y = 0$, we get $x = 2a + at_1^2$.

Therefore the normal at $(at_1^2, 2at_1)$ cuts the axis in a point whose distance from the vertex is $(2a + at_1^2)$. Similarly for the other two points.

By the given condition, $(2a + at_1^2) + (2a + at_3^2) = 2(2a + at_2^2)$

$$\text{or,} \quad t_1^2 + t_3^2 = 2t_2^2.$$

Eliminating t_1, t_2, t_3, we get $\left(\dfrac{3k}{2a - h}\right)^3 = -\dfrac{2k}{a}$.]

24. The normals at three points P, Q, R on the parabola $y^2 = 4ax$ meet at the point (h, k). Prove that the centre of gravity of the triangle PQR lies on the axis at a distance $\frac{2}{3}(h - 2a)$ units from the vertex, while the orthocentre is at the point $(h - 6a, -\frac{1}{2}k)$.

25. If the tangent and the normal to an ellipse meet the major axis at the points T and G respectively, then prove that $CG \cdot CT = CS^2$, where C is the centre and S is the nearer focus to T. [C. H. 1960]

26. QR is a chord of the equilateral hyperbola $xy = c^2$ which is a normal at Q. Show that $3CQ^2 + CR^2 = QR^2$, where C is the centre of the hyperbola. [C. H. 1971 ; N. B. H. 1991]

27. If the normal to an equilateral hyperbola at a variable point P meets the curve again in Q, then show that PQ varies as CP^3, where C is the centre of the curve. [C. H. 1973 ; N. B. H. 1990]

28. The normal at the point $P\left(ct, \frac{c}{t}\right)$ of the equilateral hyperbola $xy = c^2$ meets the hyperbola again in Q. If R be the point $\left(-ct, -\frac{c}{t}\right)$ and the circle on QR as diameter touches the axis $y = 0$, then show that

$$t^4 = \frac{1}{3}.$$

[C. H. 1975]

29. The normal to the equilateral hyperbola $xy = c^2$ at a point P on it meets the curve again at Q and touches the conjugate hyperbola. Show that

$$PQ^4 = 512c^4.$$

[C. H. 1977]

30. Show that the locus of the feet of the normals drawn from a point to the parabola $y^2 = 4ax$ is an equilateral hyperbola.

31. (*a*) If the normals at the four points $(x_i, y_i), i = 1, 2, 3, 4$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be concurrent, then prove that

$$(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4.$$

(*b*) If the normals at the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) on the rectangular hyperbola $xy = c^2$ meet at (α, β), then show that $\alpha = x_1 + x_2 + x_3 + x_4$, $\beta = y_1 + y_2 + y_3 + y_4$ and $x_1x_2x_3x_4 = y_1y_2y_3y_4 = -c^4$.

32. (*a*) Show that the sub-tangent at a point on a parabola is bisected at the vertex.

(*b*) Show that the sub-normal at any point of a parabola is equal to its semi-latus rectum.

(*c*) Show that the sub-tangent and the sub-normal of a point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $\left(\frac{a^2}{x_1} - x_1\right)$ and $\frac{b^2}{a^2}x_1$ respectively.

Answers

1. (*i*) $2x + 3y = 11$. (*ii*) $28x - 3y - 16 = 0$. (*iii*) $x + y = 2$.

5. $\left[\frac{a^2h}{a^2 - b^2}, \pm \frac{b}{a^2 - b^2}\sqrt{(a^2 - b^2)^2 - a^2h^2}\right]$.

6. $(\frac{1}{2}, 2)$. **9.** $(6, -4\sqrt{3})$.

11 POLAR EQUATIONS

11.1. Polar co-ordinates.

Let O be a fixed point called the *origin* or the *pole* and OX be a fixed straight line called the *initial line* or the *polar axis*.

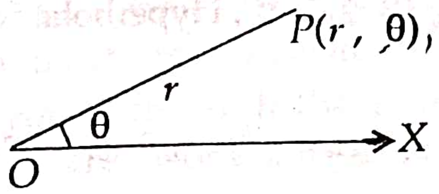

Fig. 38

Let P be any point in the plane ; OP is drawn. Let it be of length r and the angle $\angle XOP$ be θ. The length r is called the *radius vector* and the angle θ is called the *vectorial angle* of the point P. If these two elements be given, the position of the point is determined. These are called the *polar co-ordinates* of the point and are represented as $P(r, \theta)$. The radius vector is positive, if it be measured from the pole along the line bounding the vectorial angle; it is negative, if measured in the opposite direction. Vectorial angle is generally assumed positive, if measured in the anti-clock-wise direction.

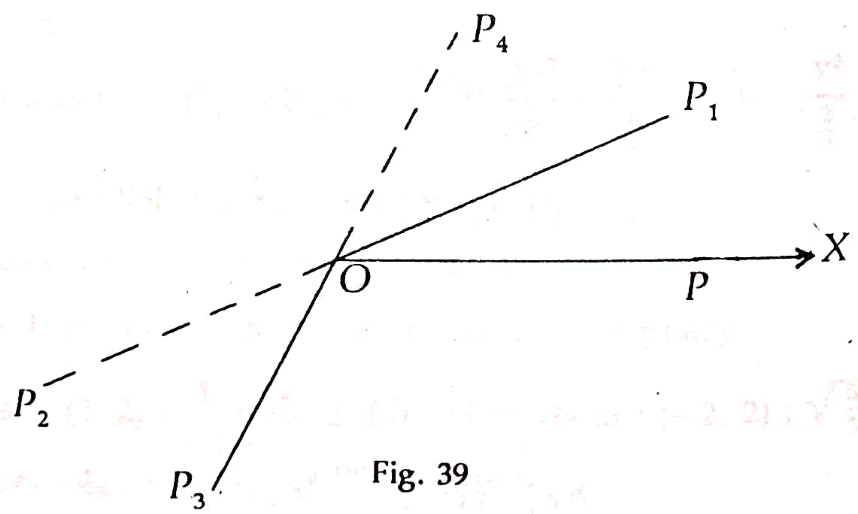

Fig. 39

Let the line segment OP of length 2 units revolve from the initial line OX through an angle of 30° in the anti-clock-wise direction to its new position OP_1. Then the polar co-ordinates of P_1 will be (2, 30°). The point P_2, situated on P_1O produced such that the lengths of

OP_1 and OP_2 are equal, will have its co-ordinates (– 2, 30°). In the same way let OP_3 be the new position of OP after revolving from OX through an angle of 120° in the clock-wise direction. Then the polar co-ordinates of P_3 will be (2, – 120°). The point P_4 situated on P_3O produced such that the lengths of OP_3 and OP_4 are equal, will have its co-ordinates (– 2, – 120°).

It can easily be seen that the same point P_4 may be denoted by each of the following four sets of polar co-ordinates :

$$(2, 60°), (2, -300°), (-2, 240°), (-2, -120°).$$

In general, the polar co-ordinates of the same point may be expressed by each of

$$(r, \theta), \{r, -(360° - \theta)\}, (-r, 180° + \theta) \text{ and } \{-r, -(180° - \theta)\}.$$

11.2. Change from cartesian to polar system of co-ordinates and vice-versa.

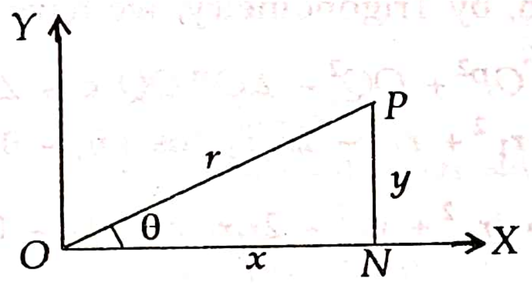

Fig. 40

Let P be any point whose cartesian co-ordinates referred to rectangular axes OXY are (x, y) and whose polar co-ordinates referred to O as pole and OX as initial line are (r, θ). Draw PN perpendicular to OX, so that we have

$$ON = x, \ NP = y,$$

$$\angle NOP = \theta \text{ and } OP = r.$$

From the triangle NOP, we have

$$x = ON = OP \cos \angle NOP = r \cos \theta, \quad \ldots (1)$$

$$y = NP = OP \sin \angle NOP = r \sin \theta, \quad \ldots (2)$$

$$r = OP = \sqrt{ON^2 + NP^2} = \sqrt{x^2 + y^2} \quad \ldots (3)$$

and

$$\tan \theta = \frac{NP}{ON} = \frac{y}{x}; \quad \ldots (4)$$

Equations (1) and (2) express the cartesian co-ordinates of P in terms of its polar co-ordinates and equations (3) and (4) express the polar co-ordinates in terms of the cartesian co-ordinates.

11.3. Distance between two points.

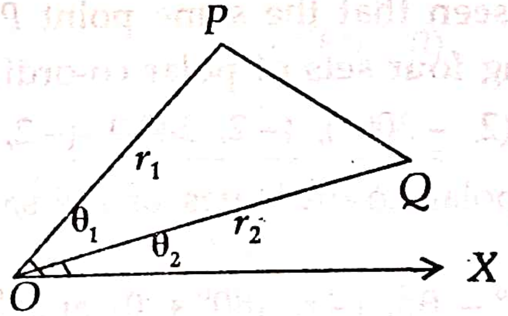

Fig. 41

Let the polar co-ordinates of the two points P and Q be (r_1, θ_1) and (r_2, θ_2) so that $OP = r_1, OQ = r_2, \angle XOP = \theta_1$, $\angle XOQ = \theta_2$. Then, by Trigonometry, we have in the triangle OPQ,

$$PQ^2 = OP^2 + OQ^2 - 2.OP.OQ \cos \angle QOP$$
$$= r_1{}^2 + r_2{}^2 - 2r_1 r_2 \cos (\theta_1 - \theta_2).$$

Therefore $PQ = \{r_1{}^2 + r_2{}^2 - 2r_1r_2 \cos (\theta_1 - \theta_2)\}^{\frac{1}{2}}$.

11.4. Area of a triangle.

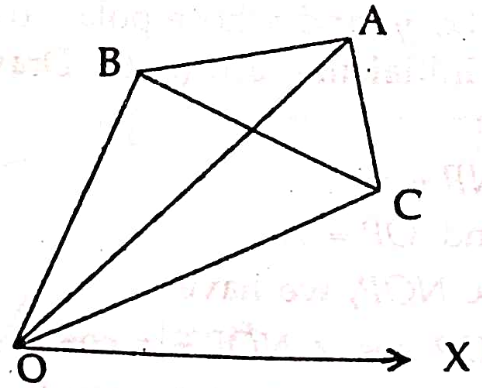

Fig. 42

Let (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) be the polar co-ordinates of the vertices A, B, C of the triangle ABC. Then

$$\Delta ABC = \Delta AOB + \Delta AOC - \Delta BOC.$$

Now $\Delta AOB = \frac{1}{2} OA.OB \sin \angle BOA$.

$$= \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1);$$

$$\Delta AOC = \frac{1}{2} OA.\ OC \sin \angle AOC$$

$$= \frac{1}{2} r_1 r_3 \sin(\theta_1 - \theta_3)$$

and $\Delta BOC = \frac{1}{2} OB.\ OC \sin \angle BOC$

$$= \frac{1}{2} r_2 r_3 \sin(\theta_2 - \theta_3)$$

$$= -\frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2).$$

Hence $\Delta ABC = \Delta AOB + \Delta AOC - \Delta BOC$

$$= \frac{1}{2}[r_2 r_1 \sin(\theta_2 - \theta_1) + r_1 r_3 \sin(\theta_1 - \theta_3) + r_3 r_2 \sin(\theta_3 - \theta_2)].$$

Cor. The area of a polygon $A_1 A_2 \ldots\ldots A_n$, the polar co-ordinates of whose vertices are $(r_1, \theta_1), (r_2, \theta_2) \ldots\ldots, (r_n, \theta_n)$ is

$$\frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2) + \cdots\cdots$$
$$\cdots\cdots + \frac{1}{2} r_n r_1 \sin(\theta_1 - \theta_n).$$

Note. If the points A, B, C be collinear, then

$$r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3) = 0.$$

11.5. Polar equation of a straight line.

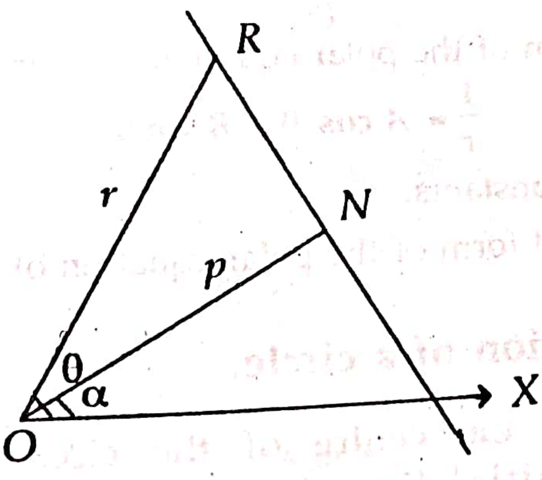

Fig. 43

Let $ON = p$ be the perpendicular on the straight line from O, the pole and α be the vectorial angle of the point N with reference to OX as the initial line. Thus N is the point (p, α).

Let $R(r, \theta)$ be any point on the straight line.

Then $\angle XOR = \theta$, so that $\angle NOR = \theta - \alpha$.

Now we have $\dfrac{ON}{OR} = \cos \angle NOR$

or, $\dfrac{p}{r} = \cos(\theta - \alpha)$, that is, $r \cos(\theta - \alpha) = p$.

This, being a relation between the polar co-ordinates of any point on the line, is the polar equation of the straight line.

Note 1. This equation may be obtained from the cartesian equation $x \cos \alpha + y \sin \alpha = p$ by puting $x = r \cos \theta$ and $y = r \sin \theta$.

Note 2. If $\alpha = 0$, the straight line becomes $p = r \cos \theta$ and is perpendicular to the polar axis; if $\alpha = \frac{1}{2}\pi$, the equation of the straight line is $p = r \sin \theta$ and is parallel to the polar axis. If $p = 0$, the straight line passes through the pole and in this case $\theta - \alpha = \frac{1}{2}\pi$, that is, $\theta =$ constant is the equation of the straight line.

Note 3. The equations of two parallel straight lines are of the form $r \cos(\theta - \alpha) = p$ and $r \cos(\theta - \alpha) = p'$. The equations of two mutually perpendicular straight lines are of the form

$$r \cos(\theta - \alpha) = p \text{ and } r \cos(\theta - \alpha') = p', \text{ where } \alpha' \sim \alpha = \tfrac{1}{2}\pi.$$

Note 4. The equation $r \cos(\theta - \alpha) = p$ may be written as

$$\cos\theta \cos\alpha + \sin\theta \sin\alpha = \frac{p}{r}$$

or, $$\frac{1}{r} = \frac{\cos\alpha}{p} \cos\theta + \frac{\sin\alpha}{p} \sin\theta.$$

Thus another form of the polar equation of a straight line is

$$\frac{1}{r} = A \cos\theta + B \sin\theta,$$

where A and B are constants.

This is the general form of the polar equation of a straight line.

11.6. Polar equation of a circle.

Let $C(R, \alpha)$ be the centre of the circle with O as the pole and OX as the initial line. Let a be the radius of the circle, $OC = R$ and $\angle XOC = \alpha$.

Let any line through O making an angle θ with the initial line meet the circle at P and Q and let $OP = r$ and $\angle XOP = \theta$. Thus the polar co-ordinates of P are (r, θ).

Then, in the triangle CPO,

$$CP^2 = OC^2 + OP^2 - 2\,OC\,.\,OP\cos\angle COP$$

or,

$$a^2 = R^2 + r^2 - 2Rr\cos(\theta - \alpha)$$

or,

$$r^2 - 2Rr\cos(\theta - \alpha) + R^2 - a^2 = 0. \quad \ldots \quad (1)$$

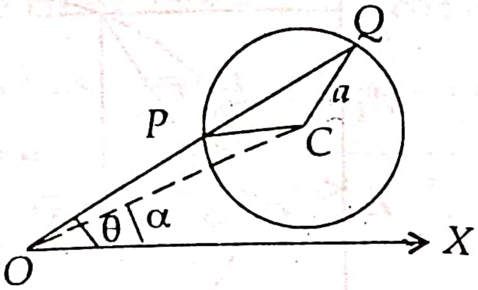

Fig. 44

(1), being a relation between the polar co-ordinates of any point on the circle, is the required polar equation of a circle of radius a.

Cor. If the initial line passes through the centre of the circle, then $\alpha = 0$ and the equation becomes $r^2 - 2Rr\cos\theta + R^2 - a^2 = 0$.

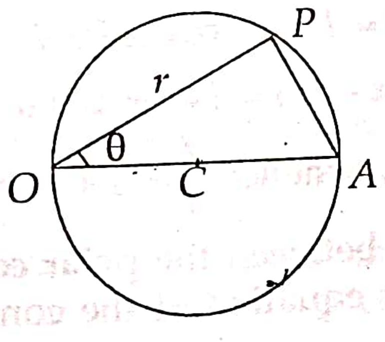

Fig. 45

If the pole be taken on the circle, then $R = a$ and the general equation reduces to $r = 2a\cos(\theta - \alpha)$.

If $R = a$ and $\alpha = \frac{1}{2}\pi$, then the polar axis becomes the tangent to the circle at the pole and the equation of the circle becomes

$$r = 2a\sin\theta.$$

11.7. Polar equation of a conic referred to a focus as pole.

Let S be the focus, XM_1M be the directrix of the conic having e for its eccentricity. Let us take the pole of the polar system at S and SX, the axis of the conic, as the initial line. Referred to the pole S and the initial line SX, let the co-ordinates of any point P on the conic be (r, θ), so that $SP = r$ and $\angle XSP = \theta$.

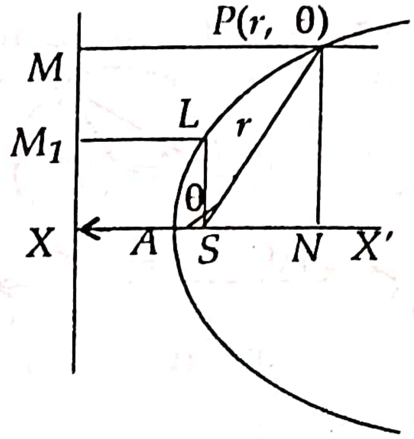

Fig. 46

PN is drawn perpendicular to the axis and SL is the semi-latus rectum.

Therefore $SL = e.LM_1 = e.SX = l$ (say).

PM is drawn perpendicular to the directrix from P.

Thus $r = SP = e.PM = e.NX = e(SX + SN)$

$= e.SX + e.SN = l + er \cos \angle PSN$

$= l + er \cos(\pi - \theta) = l - er \cos\theta$

or, $l = r + er\cos\theta$, whence $\dfrac{l}{r} = 1 + e\cos\theta$.

This, being a relation between the polar co-ordinates of any point on the conic, is the polar equation of the conic.

Note 1. If SX' be taken as the initial line so that $\angle PSN = \theta$, then the equation of the conic becomes $\dfrac{l}{r} = 1 - e\cos\theta$.

Note 2. From the equations it is evident that the curve is symmetrical about the initial line ; for, changing θ to $(-\theta)$ or to $(2\pi - \theta)$ the nature of the equation remains unaltered.

Note 3. If the axis of the conic makes an angle α with the initial line SX and $P(r, \theta)$ be any point on the conic, then

$\angle XSP = \theta - \alpha$ so that $\angle PSN = \pi - \theta + \alpha$.

Therefore the equation of the conic becomes

$$\frac{l}{r} = 1 + e \cos(\theta - \alpha).$$

If, in this case, the initial line be SX', then the equation of the conic is

$$\frac{l}{r} = 1 - e \cos(\theta - \alpha).$$

11.8. Nature of the conic $l/r = 1 + e \cos\theta$.

Case I. *If $e = 1$, then the curve is a parabola whose equation becomes*

$\frac{l}{r} = 1 + \cos\theta$, *whence* $r = \frac{l}{1 + \cos\theta} = \frac{l}{2\cos^2 \frac{1}{2}\theta}$.

At A, the vertex, $\theta = 0$, $r = \frac{1}{2}l$.

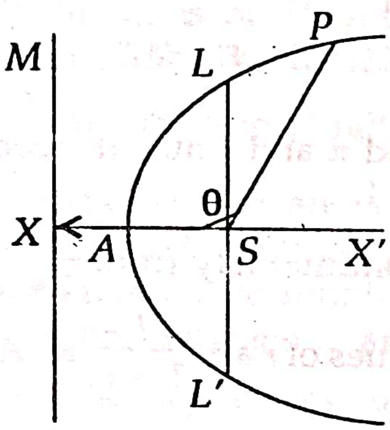

Fig. 47

When $\theta = \frac{1}{2}\pi$, $\cos\theta = 0$ and $r = l$, giving the end L of the latus rectum.

$(1 + \cos\theta)$ decreases with increase in θ, therefore r increases beyond limit as θ tends to π, since l is constant. Thus the curve extends up to infinity.

$(1 + \cos\theta)$ again will be increasing continuously as θ increases beyond π until when $\theta = \frac{3}{2}\pi$ when r becomes equal to l again, giving the other end L' of the latus rectum. Again, when $\theta = 2\pi$, $r = \frac{1}{2}l$. Thus the curve described is a parabola as shown in the figure.

Note. If SX' be taken as the initial line, the equation of the parabola becomes

$$r = \frac{l}{1 - \cos\theta} = \frac{l}{2\sin^2 \frac{1}{2}\theta} = \frac{l}{2}\operatorname{cosec}^2 \frac{\theta}{2}.$$

Case II. *If $e < 1$, then the curve is an ellipse.*

At A, $\theta = 0$ and $\frac{l}{r} = 1 + e$.

The equation of the curve being $\frac{l}{r} = 1 + e \cos\theta$, as $\cos\theta$ decreases with increase in θ, r also increases until θ reaches the value π at A' when $\frac{l}{r} = 1 - e$ and e being less than 1, r is positive.

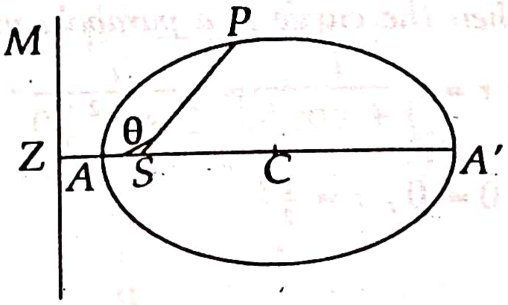

Fig. 48

As θ increases beyond π and until it reaches 2π, $\cos\theta$ goes on increasing from (-1) to 1.

Hence r decreases continuously from $\frac{l}{1-e}$ to $\frac{l}{1+e}$. So the maximum and minimum values of r are $\frac{l}{1-e}$ at A' and $\frac{l}{1+e}$ at A respectively.

Again, for any value of θ, $\cos\theta = \cos(2\pi - \theta)$, showing that the curve is symmetrical about the axis, which is the initial line.

Thus, for $e < 1$, the equation gives a closed curve symmetrical about the axis, as shown in the diagram.

Note. The lengths of the semi-axes of the ellipse are

$$\frac{l}{1 - e^2} \text{ and } \frac{l}{\sqrt{1 - e^2}}.$$

Case III. *If $e > 1$, then the curve is a hyperbola.*

At A, $\theta = 0$ and $\frac{l}{r} = 1 + e$.

When $\theta = \pi/2$, $\cos\theta = 0$ and $l = r$.

As θ increases, $\cos\theta$ decreases and hence r increases, but remains finite till $1 + e\cos\theta = 0$.

When $1 + e\cos\theta = 0$, that is, $\cos\theta = -\frac{1}{e}$, r is infinite.

If $e > 1$, then the conic has two infinite radius vectors SL and SL' corresponding to the two values of $\cos\theta = -\frac{1}{e}$.

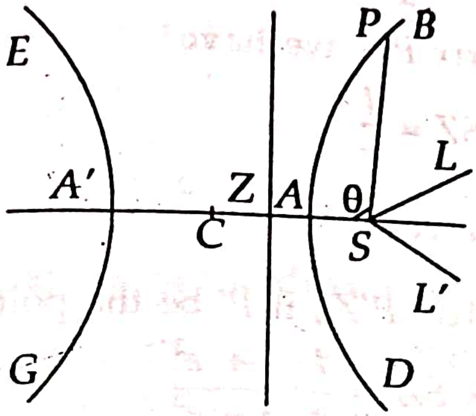

Fig. 49

Let the two values of θ be α and β which are $\angle ASL$ and $\angle ASL'$, the inclinations of SL and SL' to the initial line SA. SL and SL' are equally inclined to the initial line, that is, to the transverse axis at $\cos^{-1}\left(\frac{1}{e}\right)$ and are also parallel to the asymptotes.

If the vectorial angle θ lies between α and β, then the radius vector is negative and we get the other branch of the hyperbola. Hence a hyperbola has two branches tending to infinity, one, if θ lies outside α and β, and the other, if θ lies between α and β.

Note. The lengths of the semi-axes of the hyperbola are

$$\frac{l}{e^2 - 1} \text{ and } \frac{l}{\sqrt{e^2 - 1}}.$$

11.9. Equations of the directrices.

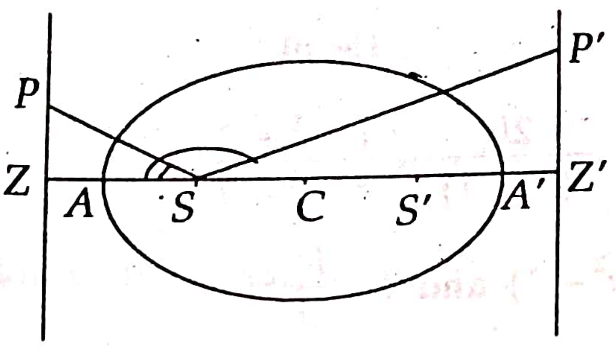

Fig. 50

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$.

If $e < 1$, the conic is an ellipse. Let the focus S be the pole and $P(r, \theta)$ be a point on the directrix ZP which is nearer to the pole.

Now $SZ = \frac{l}{e}$ and $SZ' = 2CZ - SZ = \frac{2l}{e(1-e^2)} - \frac{l}{e} = \frac{l}{e} \cdot \frac{1+e^2}{1-e^2}$,

since $b^2 = a^2(1-e^2)$, $l = \frac{b^2}{a}$ and $CZ = SZ + ae$.

Now, for the directrix PZ, we have

$$r \cos\theta = SZ = \frac{l}{e}$$

or, $$\frac{l}{r} = e \cos\theta. \quad \ldots (1)$$

For the other directrix $P'Z'$, if P' be the point (r, θ), we have

$$r \cos(\pi - \theta) = SZ' = \frac{l}{e} \cdot \frac{1+e^2}{1-e^2}$$

or, $$-r\cos\theta = \frac{l}{e} \cdot \frac{1+e^2}{1-e^2}$$

or, $$\frac{l}{r} = -\frac{1-e^2}{1+e^2} e \cos\theta. \quad \ldots (2)$$

(1) and (2) are the equations of the directrices.

If $e > 1$, then the conic is a hyperbola.

In this case, $SZ = \frac{l}{e}$ and $SZ' = SZ + 2CZ$.

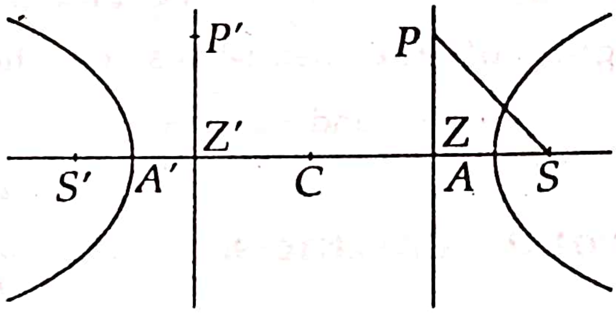

Fig. 51

Hence $SZ' = \frac{l}{e} + \frac{2l}{e(e^2-1)} = \frac{l}{e} \cdot \frac{e^2+1}{e^2-1}$,

since $b^2 = a^2(e^2-1)$ and $l = \frac{b^2}{a}$.

For the directrix PZ, we have, if the point P be (r, θ), then

$$r\cos\theta = \frac{l}{e}$$

or, $$\frac{l}{r} = e \cos\theta. \quad \ldots (3)$$

For the other directrix $P'Z'$, if P' be the point $(r, 0)$, then

$$r \cos \theta = SZ'$$

or,

$$r \cos \theta = \frac{l}{e} \cdot \frac{e^2 + 1}{e^2 - 1}$$

or,

$$\frac{l}{r} = \frac{e^2 - 1}{e^2 + 1} e \cos \theta. \qquad \ldots (4)$$

(3) and (4) are the equations of the directrices.

For a parabola, $e = 1$ and the equation of the directrix is

$$\frac{l}{r} = \cos \theta.$$

11.10. Equation of the chord of a conic.

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos \theta$. $\qquad \ldots$ (1)

Let P and Q be two points on the conic such that the radius vectors of P and Q are r_1 and r_2 and their respective vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$.

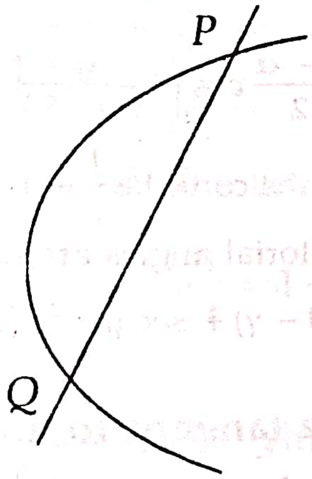

Fig. 52

Let the equation of the straight line PQ be

$$\frac{l}{r} = A \cos \theta + B \sin \theta. \qquad \ldots (2)$$

This equation contains two independent constants A and B which can be obtained from the condition that it passes through the two given points P and Q.

Now $P(r_1, \alpha - \beta)$ is a common point on the conic (1) and the straight line (2).

Therefore $\frac{l}{r_1} = 1 + e \cos(\alpha - \beta)$

and $\frac{l}{r_1} = A \cos(\alpha - \beta) + B \sin(\alpha - \beta)$.

Therefore $A \cos(\alpha - \beta) + B \sin(\alpha - \beta) = 1 + e \cos(\alpha - \beta)$

or, $(A - e) \cos(\alpha - \beta) + B \sin(\alpha - \beta) = 1$. ... (3)

Similarly, for the point $Q(r_2, \alpha + \beta)$, we have

$$(A - e) \cos(\alpha + \beta) + B \sin(\alpha + \beta) = 1. \quad \ldots (4)$$

Solving from (3) and (4), we get

$A = e + \cos\alpha \sec\beta$ and $B = \sin\alpha \sec\beta$.

Thus, from (2), the required equation of the chord is

$$\frac{l}{r} = e \cos\theta + \sec\beta \cos(\theta - \alpha). \quad \ldots (5)$$

Cor. If the equation of the conic be $\frac{l}{r} = 1 - e \cos\theta$, then the equation of the chord PQ will be

$$\frac{l}{r} = \sec\beta \cos(\theta - \alpha) - e \cos\theta.$$

Note 1. Equation of the chord joining the two points whose vectorial angles are α and β is

$$\frac{l}{r} = \sec\frac{\beta - \alpha}{2} \cos\left(\theta - \frac{\alpha + \beta}{2}\right) + e \cos\theta.$$

Note 2. If the equation of the conic be $\frac{l}{r} = 1 + e \cos(\theta - \gamma)$, then the chord joining the points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ is

$$\frac{l}{r} = e \cos(\theta - \gamma) + \sec\beta \cos(\theta - \alpha).$$

11.11. Equation of the tangent to a conic.

To find the equation of the tangent at a point whose vectorial angle is α, we are to put $\beta = 0$ in the equation of the chord joining the two points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ of the conic and we have the equation of the tangent to the conic $\frac{l}{r} = 1 + e \cos\theta$ as

$$\frac{l}{r} = e \cos\theta + \cos(\theta - \alpha).$$

Cor. If the equation of the conic be $\frac{l}{r} = 1 - e \cos\theta$, then the equation of the tangent at the point α is $\frac{l}{r} = \cos(\theta - \alpha) - e \cos\theta$.

Note. The equation of the tangent to the conic

$\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ at α is $l/r = e\cos(\theta - \gamma) + \cos(\theta - \alpha)$.

11.12. Equation of the normal to a conic.

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$, ... (1)

so that the equation of the tangent at a point whose vectorial angle is α is $\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$.

The equation of a straight line perpendicular to this line is

$$\frac{k}{r} = e\cos(\theta + \tfrac{1}{2}\pi) + \cos(\theta + \tfrac{1}{2}\pi - \alpha)$$

or,
$$\frac{k}{r} = -e\sin\theta - \sin(\theta - \alpha). \quad \ldots \quad (2)$$

Now k is so chosen that this perpendicular is the normal, that is, it passes through the point $\left(\frac{l}{1 + e\cos\alpha}, \alpha\right)$, which is the point of contact. Putting these in (2), we get

$$k\frac{1 + e\cos\alpha}{l} = -e\sin\alpha, \text{ whence } k = -\frac{le\sin\alpha}{1 + e\cos\alpha}.$$

Hence the equation of the normal is

$$\frac{le\sin\alpha}{1 + e\cos\alpha} \cdot \frac{1}{r} = e\sin\theta + \sin(\theta - \alpha).$$

Cor. If the equation of the conic be $\frac{l}{r} = 1 - e\cos\theta$, then the equation of the normal at the point α is

$$\frac{le\sin\alpha}{1 - e\cos\alpha} \cdot \frac{1}{r} = e\sin\theta - \sin(\theta - \alpha).$$

11.13. Equation of the chord of contact of tangents.

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$.

Let $(\alpha - \beta)$ and $(\alpha + \beta)$ be the vectorial angles of the two points of contact of the tangents from a given point (r_1, θ_1) to the conic. Then the equation of the chord is

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - \alpha). \quad \ldots \quad (1)$$

Now the equations of the tangents at the points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ are

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha + \beta)$$

and $$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha - \beta).$$

Both these tangents pass through the point (r_1, θ_1).

Therefore $$\left.\begin{aligned} \frac{l}{r_1} &= e \cos \theta_1 + \cos (\theta_1 - \alpha + \beta) \\ \text{and} \quad \frac{l}{r_1} &= e \cos \theta_1 + \cos (\theta_1 - \alpha - \beta). \end{aligned}\right\} \quad \ldots (2)$$

Hence $\cos (\theta_1 - \alpha + \beta) = \cos (\theta_1 - \alpha - \beta)$

or, $$\theta_1 - \alpha + \beta = 2n\pi \pm (\theta_1 - \alpha - \beta).$$

As the upper sign gives a particular value of β, we take the lower sign.

Therefore $\theta_1 - \alpha + \beta = 2n\pi - \theta_1 + \alpha + \beta$, whence $\alpha = \theta_1 - n\pi$.

From (2), $$\frac{l}{r_1} = e \cos \theta_1 + \cos (n\pi - \beta)$$
$$= e \cos \theta_1 + (-1)^n \cos \beta$$

or, $$\frac{l}{r_1} - e \cos \theta_1 = (-1)^n \cos \beta.$$

Hence, from (1), we get the required equation of the chord of contact of tangents from the point (r_1, θ_1) to the conic as

$$\left(\frac{l}{r} - e \cos \theta\right)\left(\frac{l}{r_1} - e \cos \theta_1\right) = (-1)^n \cos (\theta - \theta_1 + n\pi)$$
$$= \cos (\theta - \theta_1).$$

Cor. If the equation of the conic be $\frac{l}{r} = 1 - e \cos \theta$, then the equation of the chord of contact of the tangents to the conic from the point (r_1, θ_1) is $\left(\frac{l}{r} + e \cos \theta\right)\left(\frac{l}{r_1} + e \cos \theta_1\right) = \cos (\theta - \theta_1)$.

11.14. Equation of the polar of a point (r_1, θ_1) with respect to a conic.

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos \theta$.

Let AB be a chord passing through $P(r_1, \theta_1)$. Let the tangents at A and B meet in $T(r', \theta')$. The locus of T is the polar of P with respect

11.16. Illustrative examples.

Ex. 1. *Find the nature of the conic* $\frac{8}{r} = 4 - 5\cos\theta$.

The given equation can be written as

$$\frac{2}{r} = 1 - \frac{5}{4}\cos\theta.$$

Comparing this equation with the equation

$$\frac{l}{r} = 1 - e\cos\theta,$$

we see that here $e = \frac{5}{4} > 1$.

Hence the given equation represents a hyperbola.

Ex. 2. *Find the points on the conic* $\frac{14}{r} = 3 - 8\cos\theta$ *whose radius vector is* 2.

For the points with radius vector 2 on the given curve, we have

$$\frac{14}{2} = 3 - 8\cos\theta$$

or, $$7 = 3 - 8\cos\theta$$

or, $$\cos\theta = -\frac{4}{8} = -\frac{1}{2}.$$

Hence $\theta = \frac{2}{3}\pi, -\frac{2}{3}\pi$.

Thus the required points are $(2, \frac{2}{3}\pi)$ and $(2, -\frac{2}{3}\pi)$.

Ex. 3. *Find the polar equation of the ellipse* $\frac{x^2}{36} + \frac{y^2}{20} = 1$, *if the pole be at its right-hand focus and the positive direction of the x-axis be the positive direction of the polar axis.* [T/H. 1992]

For the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$, the lengths of the semi-major and semi-minor axes (that is, a and b) are given by $a^2 = 36$ and $b^2 = 20$.

Therefore the semi-latus rectum of the ellipse is

$$l = \frac{b^2}{a} = \frac{20}{6} = \frac{10}{3}$$

and the eccentricity e of the ellipse is given by

$$b^2 = a^2(1 - e^2)$$

or, $$20 = 36(1 - e^2)$$

or, $$1 - e^2 = \frac{20}{36} = \frac{5}{9}$$

or, $$e^2 = 1 - \frac{5}{9} = \frac{4}{9}$$

or, $$e = \frac{2}{3}.$$

The required polar equation of the ellipse is

$$\frac{l}{r} = 1 + e \cos \theta$$

or,
$$\frac{\frac{10}{3}}{r} = 1 + \frac{2}{3} \cos \theta$$

or,
$$\frac{10}{r} = 3 + 2 \cos \theta .$$

Ex. 4. *Show that the straight line $r \cos (\theta - \alpha) = p$ touches the conic $\frac{l}{r} = 1 + e \cos \theta$, if $(l \cos \alpha - ep)^2 + l^2 \sin^2 \alpha = p^2$.* [C. H. 1997]

Let the given straight line

$$r \cos (\theta - \alpha) = p$$

that is,
$$\frac{p}{r} = \cos (\theta - \alpha) = \cos \theta \cos \alpha + \sin \theta \sin \alpha \qquad \ldots (1)$$

touch the conic at the point 'β'.

The equation of the tangent to the given conic at the point 'β' is

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \beta)$$

$$= \cos \theta (e + \cos \beta) + \sin \theta \sin \beta . \qquad \ldots (2)$$

Equations (1) and (2) are identical.

Therefore
$$\frac{p}{l} = \frac{\cos \alpha}{e + \cos \beta} = \frac{\sin \alpha}{\sin \beta} .$$

Hence
$$\sin \beta = \frac{l \sin \alpha}{p} \text{ and } \cos \beta = \frac{l \cos \alpha}{p} - e .$$

Now $\sin^2 \beta + \cos^2 \beta = 1$

or,
$$\frac{l^2 \sin^2 \alpha}{p^2} + \left(\frac{l \cos \alpha}{p} - e \right)^2 = 1$$

or,
$$(l \cos \alpha - ep)^2 + l^2 \sin^2 \alpha = p^2 .$$

Ex. 5. (a) *Show that the semi-latus rectum of a conic is a harmonic mean between the segments of any focal chord.* [B. H. 1991, 2005]

(b) *If PSQ and PS'R be two chords of an ellipse through the foci S and S', then prove that $\left(\frac{SP}{SQ} + \frac{S'P}{S'R} \right)$ is independent of the position of P.*

(a) Let PSQ be a focal chord. If the vectorial angle of P be θ, then that of Q will be $(\theta + \pi)$.

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos \theta$.

Let $SP = r$ and $SQ = r'$.

Therefore

$$\frac{l}{r} = 1 + e\cos\theta \text{ and } \frac{l}{r'} = 1 + e\cos(\theta + \pi) = 1 - e\cos\theta.$$

Adding these, we get $\frac{l}{SP} + \frac{l}{SQ} = 2$, whence $\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}$.

This also shows that in any conic the sum of the reciprocals of the segments of a focal chord is constant.

(*b*) Let PSQ be the focal chord such that the vectorial angles of P and Q are α and $(\alpha + \pi)$ respectively.

Therefore $\frac{l}{SP} = 1 + e\cos\alpha$

and $\frac{l}{SQ} = 1 + e\cos(\alpha + \pi) = 1 - e\cos\alpha.$

Hence $\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}$

or, $\frac{SP}{SQ} = \frac{2}{l}\cdot SP - 1.$ $\quad\cdots$ (1)

Again $PS'R$ is also a focal chord.

Therefore $\frac{1}{S'P} + \frac{1}{S'R} = \frac{2}{l}$

or, $\frac{S'P}{S'R} = \frac{2}{l}\cdot S'P - 1.$ $\quad\cdots$ (2)

Adding (1) and (2), we get

$$\frac{SP}{SQ} + \frac{S'P}{S'R} = \frac{2}{l}(SP + S'P) - 2 = \frac{4a}{l} - 2,$$

where $2a$ is the major axis of the ellipse.

The right-hand side is independent of the position of P.

Ex. 6. *PSP' is a focal chord of the conic. Prove that the angle between the tangents at P and P' is $\tan^{-1}\frac{2e\sin\alpha}{1 - e^2}$, where α is the angle between the chord and the major axis.* [C. H. 1989, 1993]

The focal chord PSP' makes an angle α with the major axis of the conic $\frac{l}{r} = 1 + e\cos\theta$. Then the vectorial angles of P and P' are α and $(\alpha + \pi)$ respectively. The tangent at α is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta$$

or, $l = r\cos\theta(\cos\alpha + e) + r\sin\alpha\sin\theta = x(\cos\alpha + e) + y\sin\alpha.$

Therefore m = slope of the tangent at $P = -\frac{\cos\alpha + e}{\sin\alpha}$.

Similarly, m' = slope of the tangent at P'

$$= -\frac{\cos(\alpha+\pi)+e}{\sin(\alpha+\pi)} = \frac{e-\cos\alpha}{\sin\alpha}.$$

If ϕ be the angle between the tangents, then

$$\tan\phi = \frac{m-m'}{1+mm'} = \frac{2e\sin\alpha}{\sin^2\alpha - e^2 + \cos^2\alpha} = \frac{2e\sin\alpha}{1-e^2}$$

or, $\phi = \tan^{-1}\dfrac{2e\sin\alpha}{1-e^2}$.

Ex. 7. (a) *Find the point of intersection of the two tangents at α and β to the conic* $\dfrac{l}{r} = 1 + e\cos\theta$.

(b) *If the tangents at P and Q of a conic meet at a point T and S be the focus of the conic, then prove that*

$ST^2 = SP \cdot SQ$, *if the conic be a parabola.* [C. H. 1992]

(a) The tangents at α and β are

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos\theta \quad \ldots (1)$$

and $$\frac{l}{r} = \cos(\theta-\beta) + e\cos\theta. \quad \ldots (2)$$

Subtracting (2) from (1), we have

$\cos(\theta-\alpha) = \cos(\theta-\beta)$, whence $\theta-\alpha = \pm(\theta-\beta)$.

The positive sign is inadmissible ; for, in that case, $\alpha = \beta$.

Therefore $\theta - \alpha = -(\theta-\beta)$, giving $\theta = \frac{1}{2}(\alpha+\beta)$.

Substituting this value of θ in (1), we get

$$\frac{l}{r} = \cos\{\tfrac{1}{2}(\alpha+\beta)-\alpha\} + e\cos\tfrac{1}{2}(\alpha+\beta)$$

or, $\dfrac{l}{r} = \cos\frac{1}{2}(\beta-\alpha) + e\cos\frac{1}{2}(\alpha+\beta)$.

If the point of intersection of the tangents be (r_1, θ_1), then

$$\theta_1 = \tfrac{1}{2}(\alpha+\beta) \text{ and } \frac{l}{r_1} = \cos\tfrac{1}{2}(\beta-\alpha) + e\cos\tfrac{1}{2}(\alpha+\beta).$$

(b) Let the given parabola be $\dfrac{l}{r} = 1 + \cos\theta$ and let the vectorial angles of P and Q on it be α and β respectively. If S be the focus of the parabola, then

$$SP = \frac{l}{1+\cos\alpha} = \frac{l}{2}\sec^2\frac{\alpha}{2}$$

and $$SQ = \frac{l}{1+\cos\beta} = \frac{l}{2}\sec^2\frac{\beta}{2}.$$

Therefore $SP \cdot SQ = \dfrac{l^2}{4}\sec^2\dfrac{\alpha}{2}\sec^2\dfrac{\beta}{2}$. ... (1)

As in (a), the co-ordinates (r_1, θ_1) of the point of intersection T of tangents at α and β to the parabola are (since $e = 1$)

$$\theta_1 = \frac{1}{2}(\alpha + \beta) \text{ and } \frac{l}{r_1} = \cos\frac{1}{2}(\beta - \alpha) + \cos\frac{1}{2}(\alpha + \beta) = 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}.$$

Therefore $\dfrac{l}{ST} = 2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}$

or, $ST = \dfrac{l}{2}\sec\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}$

or, $ST^2 = \dfrac{l^2}{4}\sec^2\dfrac{\alpha}{2}\sec^2\dfrac{\beta}{2} = SP \, . \, SQ$, by (1).

Ex. 8. *A circle of given diameter d passes through the focus of a given conic and cuts it in four points whose distances from the focus are r_1, r_2, r_3, r_4. Prove that*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l} \text{ and } r_1 r_2 r_3 r_4 = \frac{d^2 l^2}{e^2},$$

where l is the semi-latus rectum and e is the eccentricity of the conic. [B. H. 1984]

Since the circle passes through the focus (which is the pole) of the conic whose equation is assumed to be

$$\frac{l}{r} = 1 + e\cos\theta, \quad \ldots \quad (1)$$

its equation will be $r = d\cos(\theta - \alpha)$ $\quad \ldots \quad (2)$
in which the diameter passing through the focus is inclined at an angle α to the axis.

If we eliminate θ between (1) and (2), we shall get a biquadratic equation in r, whose roots r_1, r_2, r_3, r_4 will give the distances of the points of intersection of (1) and (2) from the focus of the conic.

From (1), we have $\cos\theta = \dfrac{l - r}{er}$ and hence $\sin\theta = \sqrt{1 - \left(\dfrac{l - r}{er}\right)^2}$

Then, from (2), we have

$$r = d\cos\theta\cos\alpha + d\sin\theta\sin\alpha$$

or, $\{er^2 - dl(l - r)\cos\alpha\}^2 = \{e^2r^2 - (l - r)^2\} d^2\sin^2\alpha$

or, $e^2r^4 + 2edr^3\cos\alpha + r^2(d^2 - 2eld\cos\alpha - e^2d^2\sin^2\alpha)$
$\qquad - 2ld^2r + d^2l^2 = 0.$

Therefore $\quad r_1 r_2 r_3 r_4 = \dfrac{d^2 l^2}{e^2}.$

and $\quad \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} + \dfrac{1}{r_4} = \dfrac{\Sigma r_1 r_2 r_3}{r_1 r_2 r_3 r_4} = \dfrac{\dfrac{2ld^2}{e^2}}{\dfrac{d^2l^2}{e^2}} = \dfrac{2}{l}.$

Examples XI

1. (a) Find the rectangular cartesian co-ordinates of the points, whose polar co-ordinates are

(i) $(4, \frac{2}{3}\pi)$; (ii) $(2, \frac{7}{6}\pi)$.

(b) Find the polar co-ordinates of the points, whose cartesian co-ordinates are (i) $(-1, -1)$; (ii) $(3, -3)$.

(c) Transform the following to cartesian equations :

(i) $r = 2$; (ii) $\theta = \frac{1}{4}\pi$; (iii) $1/r = 1 + \cos\theta$.

(d) Transform the following to polar equations :

(i) $(x^2 + y^2)^2 = ax^2$; (ii) $x^3 = y^2(2a - x)$.

2. (a) Find the distance between the following points :

(i) $(1, 30°)$ and $(3, 90°)$; (ii) $(2, 40°)$ and $(4, 100°)$.

(b) Show that the locus of the point, which is always at a distance of 2 units from the point $(3, \frac{1}{3}\pi)$, is $r^2 - 6r\cos(\theta - \frac{1}{3}\pi) + 5 = 0$.

3. Find the area of the triangle, whose vertices are $A(2, -30°)$, $B(3, 120°)$ and $C(1, 210°)$.

Also find the area of the square, whose one side is AC.

4. (a) Find the polar equation of the straight line joining the two points $(1, \frac{1}{2}\pi)$ and $(2, \pi)$.

(b) Show that the polar equation of the straight line passing through the points (r_1, θ_1) and (r_2, θ_2) is

$$\frac{1}{r}\sin(\theta_1 - \theta_2) - \frac{1}{r_1}\sin(\theta - \theta_2) + \frac{1}{r_2}\sin(\theta - \theta_1) = 0.$$

Hence find the condition of collinearity of the points (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3).

(c) The vectorial angle of a point P on the straight line joining the points (r_1, θ_1) and (r_2, θ_2) is $\frac{1}{2}(\theta_1 + \theta_2)$. Find the radius vector of P.

(d) Show that the perpendicular distance of the point (r_1, θ_1) from the straight line $r\cos(\theta - \alpha) = p$ is

$$r_1\cos(\theta_1 - \theta) - p.$$

5 THE SPHERE

5.1. Equation of a sphere.

The *sphere* is a surface traced out by a moving point, which is always at a constant distance from a fixed point. The fixed point is *the centre* of the sphere and the constant distance is its *radius.*

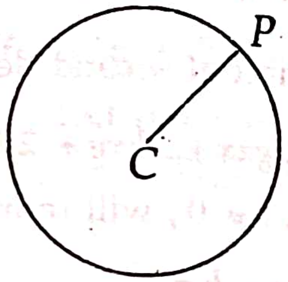

Fig. 16

Let $P(x, y, z)$ be any point on the sphere with centre at $C(x_1, y_1, z_1)$. If the radius of the sphere be r, then

$$CP^2 = r^2,$$

that is, $$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2. \quad \ldots (1)$$

This is the equation of the sphere.

If the centre be at the origin, then the equation of the sphere of radius r is
$$x^2 + y^2 + z^2 = r^2. \quad \ldots (2)$$

Note 1. The equation (1) is an equation of second degree in x, y, z in which we observe that

(*i*) the coefficients of x^2, y^2, z^2 are all equal

and (*ii*) the coefficients of yz, zx, xy are all zero.

The general equation of second degree which satisfies the above two conditions is
$$ax^2 + ay^2 + az^2 + 2lx + 2my + 2nz + d = 0.$$

On dividing by $a\,(\neq 0)$, the general equation of the sphere becomes of the form

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0. \quad (3)$$

This equation contains four constants g, f, h, c and hence a sphere can be made to satisfy four independent conditions.

In particular, a sphere may be made to pass through four non-coplanar points.

The equation (3) can be put in the form (1) as

$$(x+g)^2+(y+f)^2+(z+h)^2=g^2+f^2+h^2-c,$$

so that the centre of the sphere (3) is at $(-g, -f, -h)$ and the radius is

$$\sqrt{g^2+f^2+h^2-c}.$$

In order that the sphere may be real,

$$g^2+f^2+h^2-c \geq 0.$$

Note 2. The general equation of second degree in x, y, z, namely,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

which contains nine constants, $a \neq 0$, will represent a sphere,

$$\text{if } a=b=c \text{ and } f=g=h=0.$$

Note 3. A point lies outside, on or inside a sphere according as its distance from the centre of the sphere is greater than, equal to or less than the radius of the sphere.

5.2. Equation of a sphere on a diameter with given extremities.

Let the two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the extremities of the diameter of a sphere. Let $P(x, y, z)$ be any point on the sphere. Then PA and PB are at right angles.

The direction ratios of PA and PB are

$$(x-x_1), (y-y_1), (z-z_1) \text{ and } (x-x_2), (y-y_2), (z-z_2)$$

respectively.

Since they are at right angles,

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0,$$

which is the locus of P and hence is the equation of the sphere.

5.3. Equation of a circle.

Since the circle is a section of a sphere by a plane, the equations $x^2+y^2+z^2+2gx+2fy+2hz+d=0$ and $ax+by+cz=p$ together represent a circle.

The points, which are common to a sphere of radius r and a plane which is at a distance $p\ (<r)$ from the centre, lie on a circle of radius $\sqrt{r^2-p^2}$. This is called a *small circle*. If $p=0$, then the circle is of radius r and is called a *great circle*.

5.4. Intersection of two spheres.

The curve of intersection of two spheres is also a circle. The equations of the circle are the equations of the two spheres together.

Let $S \equiv x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0$ ... (1)

and $S' \equiv x^2 + y^2 + z^2 + 2g'x + 2f'y + 2h'z + c' = 0$ ... (2)

represent two spheres.

The co-ordinates of the points of intersection of (1) and (2) obviously satisfy the equation

$S - S' \equiv 2(g - g')x + 2(f - f')y + 2(h - h')z + c - c' = 0,$ ... (3)

which, being an equation of first degree, is the equation of the plane of intersection of the two spheres. This plane cuts either sphere in a circle.

5.5. Spheres through a given circle.

Consider the spheres

$S \equiv x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0,$ ... (1)

$S' \equiv x^2 + y^2 + z^2 + 2g'x + 2f'y + 2h'z + c' = 0$ ... (2)

and the plane

$L \equiv lx + my + nz + k = 0.$ ... (3)

(*i*) The equations (1) and (3) together represent the equations of a circle, being the intersection of $S = 0$ and $L = 0$.

The equation $S + \lambda L = 0,$ ... (4)

where λ is a constant, gives the sphere passing through the circle as given by (1) and (3) together, since (4) is satisfied by the co-ordinates of the points lying on the circle.

(*ii*) Consider the equation $S + \lambda S' = 0$, which represents a sphere passing through the circle of intersection of the spheres $S = 0$ and $S' = 0$.

In both the cases, λ is obtained from some other given conditions.

5.6. Section of a sphere with a given centre.

Let the equation of the sphere with the origin as centre be

$$x^2 + y^2 + z^2 = r^2.$$

Let $C(a, b, c)$ be the centre of the section whose equation we are to find. The line OC drawn from O to the plane through $C(a, b, c)$ is normal to the plane. The direction ratios of the normal OC are a, b, c.

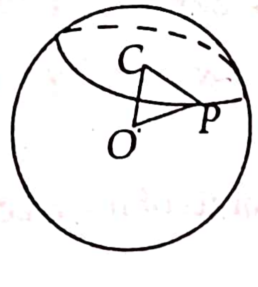

Fig. 17

Let $P(x, y, z)$ be any point on the section. Then PC, whose direction ratios are $(x-a), (y-b), (z-c)$, is perpendicular to OC.

Hence $$(x-a)a + (y-b)b + (z-c)c = 0.$$

This equation is satisfied by the co-ordinates of any point P on the plane. Hence this is the equation of the plane section whose centre is at (a, b, c).

5.7. Illustrative Examples.

Ex. 1. *Find the centre and the radius of the sphere*

$$3x^2 + 3y^2 + 3z^2 + 2x - 4y - 2z - 1 = 0.$$

Dividing the equation by 3, we have

$$x^2 + y^2 + z^2 + \tfrac{2}{3}x - \tfrac{4}{3}y - \tfrac{2}{3}z - \tfrac{1}{3} = 0$$

or, $$\left(x + \tfrac{1}{3}\right)^2 + \left(y - \tfrac{2}{3}\right)^2 + \left(z - \tfrac{1}{3}\right)^2 = \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{4}{9} + \tfrac{1}{9} = 1 = 1^2.$$

Therefore the centre is at $\left(-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)$ and the radius is 1 unit.

Ex. 2. *Find the equation of the sphere described on the join of $P(2, -3, 4)$ and $Q(-5, 6, -7)$ as diameter.*

Let $R(x, y, z)$ be any point on the sphere. Then PR and RQ are at right angles. The direction ratios of PR and RQ are

$$(x-2), (y+3), (z-4) \text{ and } (x+5), (y-6), (z+7) \text{ respectively.}$$

For perpendicularity of PR and RQ, we have

$$(x-2)(x+5) + (y+3)(y-6) + (z-4)(z+7) = 0$$

or, $$x^2 + y^2 + z^2 + 3x - 3y + 3z = 56.$$

This is the required equation of the sphere.

Ex. 3. Find the equation of the sphere which passes through the origin and touches the sphere $x^2 + y^2 + z^2 = 56$ at the point $(2, -4, 6)$.

The point $(2, -4, 6)$ lies on the given sphere whose centre is the origin. Since the required sphere passes through the origin, it touches the given sphere internally. Moreover, it is described on the segment joining the two points $(0, 0, 0)$ and $(2, -4, 6)$ as diameter. Hence the required equation of the sphere is

$$x(x-2) + y(y+4) + z(z-6) = 0$$

or,

$$x^2 + y^2 + z^2 - 2x + 4y - 6z = 0.$$

Ex. 4. Find the equation of the sphere through the points $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$.

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0. \qquad \ldots \quad (1)$$

Since the sphere passes through the given points, we have

$$d = 0,$$
$$1 + 1 + 2f - 2h + d = 0,$$
$$1 + 4 - 2g + 4f + d = 0$$

and

$$1 + 4 + 9 + 2g + 4f + 6h + d = 0.$$

Solving these equations, we get

$$g = -\tfrac{15}{14},\ f = -\tfrac{25}{14},\ h = -\tfrac{11}{14},\ d = 0.$$

Substituting these values in the equation (1), we get

$$7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0.$$

Ex. 5. *Obtain the equation of the sphere passing through four non-coplanar points (x_i, y_i, z_i), $i = 1, 2, 3, 4$.* [C. H. 1983]

Th general equation of a sphere is

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0. \qquad \ldots \quad (1)$$

If the given points lie on the sphere (1), then we must have

$$x_i^2 + y_i^2 + z_i^2 + 2gx_i + 2fy_i + 2hz_i + c = 0, \qquad \ldots \quad (2)$$
$$(i = 1, 2, 3, 4).$$

Eliminating the four constants g, f, h, c from (1) and the four equations (2), we get the required equation of the sphere as

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Note. Here the four given points are non-coplanar. So we get one and only one sphere.

If the four points be coplanar, three of them lying on a straight line, there will be no sphere through them.

In general, there can be many spheres through four coplanar points.

Ex. 6. *A plane passing through a fixed point* (a, b, c) *cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is* $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

Let the equation of the plane ABC be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1,$$

so that A is $(\alpha, 0, 0)$, B is $(0, \beta, 0)$ and C is $(0, 0, \gamma)$.

It passes through the point (a, b, c).

Therefore $$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1. \qquad \dots (1)$$

The sphere $OABC$ passing through the points $(0, 0, 0)$, $(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$ is $x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$.

If the centre of the sphere be at (x_1, y_1, z_1), then

$$x_1=\tfrac{1}{2}\alpha,\ y_1=\tfrac{1}{2}\beta,\ z_1=\tfrac{1}{2}\gamma.$$

Eliminating α, β, γ from (1), we have

$$\frac{a}{2x_1}+\frac{b}{2y_1}+\frac{c}{2z_1}=1$$

or, $$\frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=2.$$

Hence the locus of (x_1, y_1, z_1) is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

9(b) **Ex. 7.** *Find the centre and the radius of the circle*

$$x^2+y^2+z^2=25,\ x+2y+2z+9=0.$$

The centre of the sphere $x^2+y^2+z^2=25$ is at $(0, 0, 0)$ and its radius is 5 units. Length of the perpendicular from the centre $(0, 0, 0)$ of the sphere to the plane $x+2y+2z+9=0$ is

$$d=\frac{9}{\sqrt{9}}=3.$$

Therefore (radius of the circle)^2 = (radius of the sphere)$^2 - d^2 = 16$.

Hence the radius of the circle is 4 units.

Equations of the straight line perpendicular to the plane $x+2y+2z+9=0$ passing through the origin are $\frac{x}{1}=\frac{y}{2}=\frac{z}{2}=r$ (say).

Any point $(r, 2r, 2r)$ of this straight line will lie on the plane, if

$$r + 4r + 4r + 9 = 0, \text{ that is, } r = -1.$$

Hence the co-ordinates of the foot of the perpendicular, that is, the centre of the given circle is $(-1, -2, -2)$.

Ex. 8. Find the equations of the circle on the sphere $x^2 + y^2 + z^2 = 49$ *whose centre is at the point* $(2, -1, 3)$.

The equation of the plane section of the given sphere whose centre is at $(2, -1, 3)$ is

$$(x-2)2 + (y+1)(-1) + (z-3)3 = 0$$

or, $$2x - y + 3z = 14.$$

The equations of the required circle are the section of the sphere $x^2 + y^2 + z^2 = 49$ by the plane $2x - y + 3z = 14$.

✗✓Ex. 9. *Find the equation of the sphere for which the circle,* $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ *is a great circle.*

Let the equation of the sphere through the circle be

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0, \quad \ldots \quad (1)$$

λ being a constant.

Since the centre of the sphere coincides with the centre of the circle in the case of a great circle, the centre of the sphere $\{-\lambda, \frac{1}{2}(-7-3\lambda), (1-2\lambda)\}$ must lie on the plane $2x + 3y + 4z = 8$.

Therefore $2(-\lambda) + \frac{3}{2}(-7-3\lambda) + 4(1-2\lambda) = 8$, whence $\lambda = -1$.

Hence the equation of the sphere becomes

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0.$$

✓Ex. 10. *Prove that the circles*

$$2(x^2 + y^2 + z^2) + 8x - 13y + 17z - 17 = 0, \; 2x + y - 3z + 1 = 0$$

and $$x^2 + y^2 + z^2 + 3x - 4y + 3z = 0, \; x - y + 2z - 4 = 0$$

lie on the same sphere. Find its equation.

Equation of the sphere through the first circle is of the type

$$x^2 + y^2 + z^2 + 4x - \frac{13}{2}y + \frac{17}{2}z - \frac{17}{2} + \lambda(2x + y - 3z + 1) = 0 \quad \ldots \quad (1)$$

and through the second circle is of the form

$$x^2 + y^2 + z^2 + 3x - 4y + 3z + \mu(x - y + 2z - 4) = 0. \quad \ldots \quad (2)$$

If the same sphere is to contain both the given circles, then the equations (1) and (2) will be identical and thus comparing coefficients, we get

$$4+2\lambda=3+\mu, \quad \ldots (3)$$

$$\lambda-\tfrac{13}{2}=-4-\mu, \quad \ldots (4)$$

$$\tfrac{17}{2}-3\lambda=3+2\mu, \quad \ldots (5)$$

$$\lambda-\tfrac{17}{2}=-4\mu. \quad \ldots (6)$$

Equations (3) and (4) give $\lambda=\frac{1}{2}$ and $\mu=2$ and these values satisfy the relations (5) and (6). Thus, with these values of λ and μ, the equations (3), (4), (5), (6) are consistent.

Thus both the circles lie on the sphere

$$x^2+y^2+z^2+4x-\tfrac{13}{2}y+\tfrac{17}{2}z-\tfrac{17}{2}+\tfrac{1}{2}(2x+y-3z+1)=0$$

or, $x^2+y^2+z^2+5x-6y+7z-8=0$.

Examples V(A)

1. Find the equation of the sphere having
   (*i*) the centre at $(2, -3, 4)$ and radius equal to 5 units;
   (*ii*) the centre at $(-1, 2, 3)$ and diameter equal to 6 units;
   (*iii*) the centre at $(0, 0, 0)$ and passing through the point $(1, 2, 3)$.
2. Find the centre and the radius of the sphere given by
   (*i*) $2(x^2+y^2+z^2)-2x+4y-6z=15$;
   (*ii*) $x^2+y^2+z^2-2x+4y-6z=11$;
   (*iii*) $x^2+y^2+z^2-6x+8y=0$.
3. (*a*) Find the equation of the sphere passing through the four points
   (*i*) $(0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c)$;
   (*ii*) $(1, -1, -1), (3, 3, 1), (-2, 0, 5), (-1, 4, 4)$;
   (*iii*) $(0, 0, 0), (-a, b, c), (a, -b, c), (a, b, -c)$.
   Find also the centre and the radius of the sphere (*i*).

   (*b*) Obtain the equation of the sphere circumscribing the tetrahedron formed by the planes

$$x=y=z=0 \text{ and } 2x+3y+4z-12=0.$$

   (*c*) The plane $x+2y+3z=6$ meets the co-ordinate axes in A, B, C. Find the equation of the sphere $OABC$, O being the origin; and determine the centre and the radius of the sphere.

   (*d*) Find the equation of the sphere passing through the points $(3, 1, -3), (-2, 4, 1), (-5, 0, 0)$ and whose centre lies on the plane $2x+y-z+3=0$.

(e) Show that the equation of the sphere passing through the points $(0, -2, -4)$, $(2, -1, -1)$ and having its centre on the straight line $2x - 3y = 0 = 5y + 2z$ is $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$.

(f) Find the equation of the sphere, which passes through the points $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$ and has the least possible radius.

[Let the equation of the sphere be $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

Then $r^2 = (2 - a)^2 + b^2 + c^2 = a^2 + (2 - b)^2 + c^2 = a^2 + b^2 + (2 - c)^2$.

Therefore $a = b = c$ and $r^2 = 3\left\{\left(a - \frac{2}{3}\right)^2 + \frac{8}{9}\right\}$. For least value of r, $a = \frac{2}{3}$.]

4. Is there a sphere passing through the points $(1, 2, 3)$, $(4, 0, 1)$, $(-5, 6, -11)$ and $(10, -4, 9)$?

5. (a) Show that the equation of a circle passing through the points $(2, -1, -3)$, $(1, 1, -3)$ and $(-1, 5, 0)$ is given by

$$x^2 + y^2 + z^2 - 3x - 7z - 29 = 0,\ 2x + y - 3 = 0.$$

[The required circle is the intersection of any sphere passing through the three given points by the plane through these three points.]

(b) The plane ABC, whose equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, meets the axes in A, B, C. Show that the equation of the circum-circle of ΔABC is given by $x^2 + y^2 + z^2 - ax - by - cz = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$,

whose centre is $\left(\frac{a}{2} \cdot \frac{b^{-2} + c^{-2}}{k}, \frac{b}{2} \cdot \frac{c^{-2} + a^{-2}}{k}, \frac{c}{2} \cdot \frac{a^{-2} + b^{-2}}{k}\right)$,

where $k = a^{-2} + b^{-2} + c^{-2}$.

6. (a) (i) Find the centre and the radius of the circle

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0,\ x + 2y + 2z = 15.$$

(ii) Find the radius of the circle

$$3x^2 + 3y^2 + 3z^2 + x - 5y - 2 = 0,\ x + y = 2.$$

(iii) Find the centre and the radius of the circle

$$(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 100,\ 2x - 2y - z + 9 = 0.$$

(b) Show that the circle in which the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z = 2$ is cut by the plane $x + 2y + 2z = 20$ has its centre at the point $(2, 4, 5)$ and radius $\sqrt{7}$ units.

(c) If r be the radius of the circle

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ lx + my + nz = 0,$$ then

Prove that $(r^2 + d)(l^2 + m^2 + n^2) = (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2$ and find the centre. [C. H. 2005]

9. (a) (i) $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0.$

(ii) $5(x^2 + y^2 + z^2) - 18x - 27y - 36z = 0.$

(b) $5(x^2 + y^2 + z^2) + 28x - 44y - 26z + 32 = 0.$

(c) $5(x^2 + y^2 + z^2) - 44x - 38y - 14z - 48 = 0.$

10. $x^2 + y^2 + z^2 \pm 6z - 4 = 0.$

15. (b) $P(-2, -2, 7).$

16. (a) (i) $9(x^2 + y^2 + z^2) + 2x + 26y - 34z + 13 = 0.$

(ii) $x^2 + y^2 + z^2 + 2x + 2y + 2z = 3.$

(b) $9(x^2 + y^2 + z^2) - 10x + 20y - 20z - 31 = 0;$

$\left(\frac{5}{9}, -\frac{10}{9}, \frac{10}{9}\right), \frac{2}{3}\sqrt{14}$ units.

5.8. Intersection of a straight line and a sphere.

Let the equations of the sphere and the straight line be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0 \quad \ldots (1)$$

and $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots (2)$$

Any point on the line (2) will have co-ordinates

$$(\alpha + lr, \beta + mr, \gamma + nr).$$

If this point be also on the sphere, then we have

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2g(\alpha + lr) + 2f(\beta + mr) + 2h(\gamma + nr) + d = 0$$

or, $$r^2(l^2 + m^2 + n^2) + 2r\{l(\alpha + g) + m(\beta + f) + n(\gamma + h)\} + \alpha^2 + \beta^2 + \gamma^2 + 2g\alpha + 2f\beta + 2h\gamma + d = 0. \quad \ldots (3)$$

This equation, being a quadratic in r, gives two values of r, which shows that the line intersects the sphere at two points. These two points will be real and distinct, real and coincident or imaginary according as the roots of the equation (3) are real and distinct, real and equal or imaginary.

If l, m, n be the actual direction cosines of the line, then the two values of r as given by (3), putting $l^2 + m^2 + n^2 = 1$, will give the distances of the points of intersection from the point (α, β, γ).

If the two points of intersection be coincident, then the line is called the *tangent line* to the sphere, the point of contact of the tangent being

$$(\alpha + lr, \beta + mr, \gamma + nr).$$

If a secant line be drawn through A to intersect the sphere S at P and Q, then the quantity $AP.AQ$ is called the *power* of A with respect to the sphere S. In the present case, if A be the point (α, β, γ), then

$AP.AQ$ = product of the roots of the equation (3)

$$= \alpha^2 + \beta^2 + \gamma^2 + 2g\alpha + 2f\beta + 2h\gamma + d,$$

which is a constant, independent of l, m, n.

5.9. Equation of the tangent plane.

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0. \quad \ldots \quad (1)$$

Equations of any straight line through the point (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \quad \ldots \quad (2)$$

If the point (α, β, γ) lies on the sphere, then

$$\alpha^2 + \beta^2 + \gamma^2 + 2g\alpha + 2f\beta + 2h\gamma + d = 0. \quad \ldots \quad (3)$$

Any point on the line (2) is $(\alpha + lr, \beta + mr, \gamma + nr)$.

This will lie on the sphere (1), if

$$(l^2 + m^2 + n^2) r^2 + 2(\alpha l + \beta m + \gamma n + lg + mf + nh) r$$
$$+ (\alpha^2 + \beta^2 + \gamma^2 + 2g\alpha + 2f\beta + 2h\gamma + d) = 0$$

or, $(l^2 + m^2 + n^2) r^2 + 2(\alpha l + \beta m + \gamma n + lg + mf + nh) r = 0, \quad \ldots \quad (4)$

by (3).

One root of this equation is zero, which shows that one of the points of intersection coincides with the point (α, β, γ).

In order that the line (2) is a tangent line to (1) at (α, β, γ), the other point of intersection should also coincide with (α, β, γ), that is, the other root of the equation should also vanish. This requires that

$$l(\alpha + g) + m(\beta + f) + n(\gamma + h) = 0. \quad (5)$$

Now, $(-g, -f, -h)$ being the co-ordinates of the centre of the sphere, $(\alpha + g)$, $(\beta + f)$, $(\gamma + h)$ are the direction ratios of the line joining the centre and the point of contact while l, m, n are the direction numbers of the line. The condition (5) implies that if the line (2) be a tangent line to the sphere, then the line joining the centre to the point of contact is perpendicular to the tangent line.

The *tangent plane* at (α, β, γ) is the locus of all such tangent lines and is obtained by eliminating l, m, n between (5) and the line. Thus the equation of the tangent plane at the point (α, β, γ) to the sphere (1) is $(x - \alpha)(\alpha + g) + (y - \beta)(\beta + f) + (z - \gamma)(\gamma + h) = 0$

or, $x(\alpha + g) + y(\beta + f) + z(\gamma + h) - (\alpha^2 + \beta^2 + \gamma^2 + g\alpha + f\beta + h\gamma) = 0$

or, $x(\alpha + g) + y(\beta + f) + z(\gamma + h) + g\alpha + f\beta + h\gamma + d = 0$, by (3).

or, $x\alpha + y\beta + z\gamma + g(x + \alpha) + f(y + \beta) + h(z + \gamma) + d = 0$.

Cor. If the equation of the sphere be $x^2 + y^2 + z^2 = a^2$, then $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ will be a tangent line to the sphere, if

$$l\alpha + m\beta + n\gamma = 0.$$

Furthermore, the equation of the tangent plane at (α, β, γ) is

$$x\alpha + y\beta + z\gamma = a^2.$$

Note. There does not exist any tangent to a sphere at points which are inside the sphere ; because, for inside point (α, β, γ), the roots of the equation (4) are not real.

5.10. Condition of tangency of a plane.

Let the equation of the plane be $lx + my + nz = p$ ... (1)

which touches the sphere

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0. \quad \ldots (2)$$

If the plane touches the sphere, then the length of the perpendicular from the centre of the sphere to the plane will be equal to the radius of the sphere.

Now the centre of the sphere is at $(-g, -f, -h)$ and the radius is $\sqrt{g^2 + f^2 + h^2 - d}$.

Hence the required condition is

$$\left| \frac{-lg - mf - nh - p}{\sqrt{l^2 + m^2 + n^2}} \right| = \sqrt{g^2 + f^2 + h^2 - d}.$$

Squaring and rearranging, we get the condition of tangency as

$$(gl + fm + hn + p)^2 = (l^2 + m^2 + n^2)(g^2 + f^2 + h^2 - d). \quad \ldots (3)$$

Let us now suppose that the plane $lx + my + nz = p$ is such that the equation (3) holds.

Then we have $gl + fm + hn + p \neq 0$. ... (4)

Let us now consider the following system of linear equations in four variables x, y, z, t :

$$gx + fy + hz + pt = -d,$$
$$x + 0.y + 0.z - lt = -g,$$
$$0.x + y + 0.z - mt = -f,$$
$$0.x + 0.y + z - nt = -h.$$

The determinant formed by the coefficients of the variables is

$$\begin{vmatrix} g & f & h & p \\ 1 & 0 & 0 & -l \\ 0 & 1 & 0 & -m \\ 0 & 0 & 1 & -n \end{vmatrix} = g\begin{vmatrix} 0 & 0 & -l \\ 1 & 0 & -m \\ 0 & 1 & -n \end{vmatrix} - \begin{vmatrix} f & h & p \\ 1 & 0 & -m \\ 0 & 1 & -n \end{vmatrix}$$

$$= -(gl + fm + hn + p) \neq 0, \text{ by (4).}$$

Hence the system of equations will have a unique solution and let it be (x_1, y_1, z_1, t_1),

so that
$$gx_1 + fy_1 + hz_1 + d = -pt_1,$$
$$x_1 + g = lt_1,$$
$$y_1 + f = mt_1,$$
$$z_1 + h = nt_1.$$

Hence $\dfrac{x_1 + g}{l} = \dfrac{y_1 + f}{m} = \dfrac{z_1 + h}{n} = \dfrac{gx_1 + fy_1 + hz_1 + d}{-p}$.

This shows that the equations

$$(x_1 + g)x + (y_1 + f)y + (z_1 + h)z + gx_1 + fy_1 + hz_1 + d = 0$$

and $lx + my + nz - p = 0$

are identical.

From this, we can conclude that under the condition (3) the plane $lx + my + nz = p$ is a tangent plane to the given sphere. Thus (3) is the necessary and sufficient condition for tangency.

5.11. The plane of contact of the tangent planes to a sphere.

The plane containing the locus of the points of contact of the tangent planes which pass through a given outside point with respect to a sphere is the *plane of contact* of the tangent planes of the sphere.

In the case of a sphere, the locus is a circle.

Let the given point be (α, β, γ) and the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0.$$

Let (x', y', z') be a point on the sphere, so that the tangent plane passes through the point (α, β, γ).

The tangent plane at the point (x', y', z') on the sphere is

$$xx' + yy' + zz' + g(x + x') + f(y + y') + h(z + z') + d = 0.$$

This passes through the point (α, β, γ).

$$\therefore \alpha x' + \beta y' + \gamma z' + g(\alpha + x') + f(\beta + y') + h(\gamma + z') + d = 0.$$

Hence the locus of (x', y', z') is

$$\alpha x + \beta y + \gamma z + g(x + \alpha) + f(y + \beta) + h(z + \gamma) + d = 0.$$

This is the plane of contact of the tangent planes from the point (α, β, γ).

5.12. The polar plane of a point with respect to a sphere.

The locus of the intersection of the tangent planes drawn at the extremities of chords passing through a point to the sphere is called the *polar plane* of the point with respect to the sphere.

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0. \quad \ldots (1)$$

Let (x', y', z') be the point of intersection of the tangent planes at the extremities of the chord passing through the point (α, β, γ).

The plane of contact of the tangents from (x', y', z') to the sphere (1) is $xx' + yy' + zz' + g(x + x') + f(y + y') + h(z + z') + d = 0.$

This plane passes through the point (α, β, γ).

$\therefore \ \alpha x' + \beta y' + \gamma z' + g(\alpha + x') + f(\beta + y') + h(\gamma + z') + d = 0.$

Thus the locus of the point (x', y', z') is

$$\alpha x + \beta y + \gamma z + g(x + \alpha) + f(y + \beta) + h(z + \gamma) + d = 0. \quad \ldots (2)$$

This is the polar plane of the point (α, β, γ) with respect to the given sphere (1).

The point (α, β, γ) is called the *pole* of the plane (2) with respect to the sphere (1).

5.13. Pole of a plane with respect to a sphere.

To find the pole of the plane $lx + my + nz = p$ with respect to the sphere

$$x^2 + y^2 + z^2 = a^2.$$

Let (α, β, γ) be the pole of the plane $lx + my + nz = p. \quad \ldots (1)$

The polar plane of the point (α, β, γ) with respect to the given sphere is

$$\alpha x + \beta y + \gamma z = a^2. \quad \ldots (2)$$

Comparing (1) and (2), we get

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{a^2}{p}, \text{ giving } \alpha = \frac{a^2 l}{p}, \ \beta = \frac{a^2 m}{p}, \ \gamma = \frac{a^2 n}{p}.$$

Hence the pole is at $\left(\frac{la^2}{p}, \frac{ma^2}{p}, \frac{na^2}{p}\right)$.

5.14. Length of the tangent to a sphere.

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0.$$

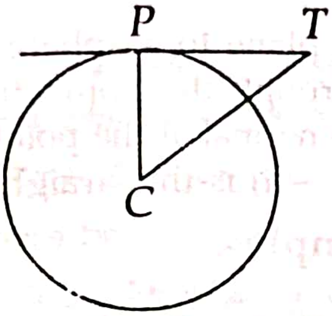

Fig. 18

Let PT be the tangent line to the sphere from the point (α, β, γ). The centre of the sphere is at $(-g, -f, -h)$. Then the angle TPC is a right angle. Hence

$$PT^2 = TC^2 - CP^2$$
$$= (\alpha + g)^2 + (\beta + f)^2 + (\gamma + h)^2 - (g^2 + f^2 + h^2 - d).$$

Length of the tangent PT

$$= \sqrt{\alpha^2 + \beta^2 + \gamma^2 + 2g\alpha + 2f\beta + 2h\gamma + d}. \quad \ldots \quad (1)$$

Thus the *power* of a point with respect to a sphere is equal to the square of the length of the tangent from the point to the sphere.

Note. The point T is outside, on or inside the sphere according as the expression under the radical sign in (1) is positive, zero or negative.

5.15. Equation of the normal at a point.

The *normal* to a sphere at the point (x_1, y_1, z_1) is the straight line through the point and perpendicular to the tangent plane to the sphere at the point (x_1, y_1, z_1).

Let the equation of the sphere be $\quad \ldots \quad (1)$

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0.$$

Its tangent plane at the point (x_1, y_1, z_1) is

$$xx_1 + yy_1 + zz_1 + g(x + x_1) + f(y + y_1) + h(z + z_1) + d = 0$$

or, $(x_1 + g)x + (y_1 + f)y + (z_1 + h)z + (gx_1 + fy_1 + hz_1 + d) = 0.$

The direction ratios of the normal to this plane are

$$(x_1 + g),\ (y_1 + f),\ (z_1 + h).$$

Hence the equations of the normal to the sphere (1) at the point (x_1, y_1, z_1) are

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f} = \frac{z - z_1}{z_1 + h}.$$

Note. Since the tangent plane to a sphere at a point is perpendicular to the radius of the sphere through the point, this radius is the normal to the sphere at the point. Thus the normal at the point $P(x_1, y_1, z_1)$ to the sphere (1) whose centre is $C(-g, -f, -h)$ is the straight line CP.

5.16. Illustrative Examples.

Ex. 1. *Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 49$ at the point $(6, -3, -2)$. Show further that $2x - 6y + 3z - 49 = 0$ is a tangent plane to the same sphere. Find the point of contact.*

The equation of the required tangent plane is

$$6x - 3y - 2z = 49.$$

The centre of the sphere is the origin (0, 0, 0). Hence the length of the perpendicular from the centre to the plane $2x - 6y + 3z - 49 = 0$ is

$$\left| \frac{-49}{\sqrt{4 + 36 + 9}} \right|, \text{ that is, } 7,$$

which is equal to the radius of the sphere.

Hence the given plane touches the sphere $x^2 + y^2 + z^2 = 49$.

Equations of the straight line through the centre (0, 0, 0) and perpendicular to the plane $2x - 6y + 3z - 49 = 0$ are

$$\frac{x}{2} = \frac{y}{-6} = \frac{z}{3}.$$

Any point on this straight line is $(2r, -6r, 3r)$.

This point lies on the plane $2x - 6y + 3z - 49 = 0$, if

$$4r + 36r + 9r = 49$$

that is, if $r = 1$.

Hence the point of contact is $(2, -6, 3)$.

Ex. 2. *Find the values of c for which the plane $x + y + z = c$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.*

The centre of the sphere is at the point (1, 1, 1) and its radius is

$$\sqrt{1 + 1 + 1 + 6} = 3.$$

The plane will touch the sphere, if the perpendicular distance of the point (1, 1, 1) from the plane be equal to the radius of the sphere.

In other words,

$$\left|\frac{1+1+1-c}{\sqrt{1+1+1}}\right| = 3$$

or, $(3-c)^2 = 27$

or, $c^2 - 6c - 18 = 0,$

which gives $c = 3(1 \pm \sqrt{3})$.

Ex. 3. Find the equation of the sphere, which passes through the points (1,0,0), (0,1,0), (0,0,1) and which touches the plane

$$2x + 2y - z = 15.$$

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0. \quad \ldots \quad (1)$$

As it passes through the points (1,0,0), (0,1,0) and (0,0,1), we have

$$1 + 2g + d = 0, \; 1 + 2f + d = 0 \text{ and } 1 + 2h + d = 0.$$

Therefore $g = f = h = -\frac{1}{2}(1+d)$. $\quad \ldots \quad (2)$

Since the plane $2x + 2y - z = 15$ touches the sphere (1), the distance of the centre $(-g, -f, -h)$ from the plane is equal to its radius which is

$$\sqrt{g^2 + f^2 + h^2 - d}.$$

Hence $\left|\frac{-2g - 2f + h - 15}{\sqrt{4+4+1}}\right| = \sqrt{g^2 + f^2 + h^2 - d}$

or, $(2g + 2f - h + 15)^2 = 9(g^2 + f^2 + h^2 - d)$

or, $(g+5)^2 = 3g^2 - d$, by (2)

or, $2g^2 - 8g - 24 = 0$

or, $(g-6)(g+2) = 0$, giving $g = 6$ or -2.

Therefore $g = f = h = 6$ or -2.

Hence $d = -13$ or 3.

Thus the equation of the sphere is

$$x^2 + y^2 + z^2 + 12x + 12y + 12z - 13 = 0$$

or $$x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0.$$

Ex. 4. If a sphere touches the planes

$$2x + 3y - 6z + 14 = 0 \text{ and } 2x + 3y - 6z + 42 = 0$$

and if its centre lies on the straight line $2x + z = 0$, $y = 0$, find the equation of the sphere.

[C. H. 1998]

The distance between the two given parallel planes is 4 units which is equal to the diameter of the sphere. Hence its radius is 2 units.

Let the centre of the required sphere be (α, β, γ). Since it lies on the straight line $2x + z = 0$, $y = 0$, we have

$$2\alpha + \gamma = 0, \; \beta = 0. \quad \ldots \quad (1)$$

Now the distance of the centre $(\alpha, 0, \gamma)$ from the plane $2x + 3y - 6z + 42 = 0$ must be equal to the radius of the sphere. Thus

$$\left| \frac{2\alpha - 6\gamma + 42}{\sqrt{4 + 9 + 36}} \right| = 2$$

or, $\quad 2\alpha - 6\gamma + 42 = 14$

or, $\quad 2\alpha - 6\gamma = -28.$ ... (2)

From (1) and (2), we have $\alpha = -2, \gamma = 4$

Thus the equation of the required sphere is

$$(x + 2)^2 + y^2 + (z - 4)^2 = 4.$$

Ex. 5. *Find the equation of the sphere touching the three co-ordinate planes.*

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0.$$

The centre of the sphere is $(-g, -f, -h)$ and its radius is equal to

$$\sqrt{g^2 + f^2 + h^2 - d}.$$

If the sphere touches the yz-plane, that is, $x = 0$, then the condition of tangency gives

$$-g = \sqrt{g^2 + f^2 + h^2 - d}$$

or, $\quad f^2 + h^2 = d.$

Similarly, applying the condition of tangency to the planes $y = 0$ and $z = 0$, we get respectively

$$h^2 + g^2 = d \text{ and } g^2 + f^2 = d.$$

Adding these three, we get

$$g^2 + f^2 + h^2 = \frac{3}{2}d.$$

Therefore $\quad g^2 = f^2 = h^2 = \frac{1}{2}d = a^2 \text{ (say)}.$

Thus $\quad g = f = h = \pm a$

and the radius $= \sqrt{g^2 + f^2 + h^2 - d} = a.$

Hence the required equation of the sphere touching the co-ordinate planes is

$$x^2 + y^2 + z^2 \pm 2ax \pm 2ay \pm 2az + 2a^2 = 0.$$

Note. There can be an infinite number of such spheres depending on the value of a. For some particular value of a there may be eight such spheres.

Ex. 6. *Show that only one tangent plane can be drawn to the sphere $x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0$ through the straight line*

$$3x - 4y - 8 = 0 = y - 3z + 2.$$

Find the equation of the plane. [*T. H. 1991 ; C. H. 1996 ; V. H. 1997*]

The equation of the plane through the given straight line is of the form

$$\lambda(3x - 4y - 8) + \mu(y - 3z + 2) = 0, \quad \ldots \quad (1)$$

where λ and μ are variable parameters and both of them are not simultaneously zero.

This equation of the plane is

$$3\lambda x + (\mu - 4\lambda)y - 3\mu z = 8\lambda - 2\mu. \quad \ldots \quad (2)$$

The plane (2) will touch the given sphere, if

$$\{-3\lambda + 3(\mu - 4\lambda) - 3\mu + 8\lambda - 2\mu\}^2 = \{9\lambda^2 + (\mu - 4\lambda)^2 + 9\mu^2\}(1 + 9 + 1 - 8)$$

or, $(-7\lambda - 2\mu)^2 = 3(25\lambda^2 + 10\mu^2 - 8\lambda\mu)$

or, $26\lambda^2 + 26\mu^2 - 52\lambda\mu = 0$.

This gives $\lambda^2 - 2\lambda\mu + \mu^2 = 0$, that is, $\left(\frac{\lambda}{\mu} - 1\right)^2 = 0$, which gives only one value of $\frac{\lambda}{\mu}$.

Therefore $\frac{\lambda}{\mu} = 1$.

Hence there is only one tangent plane, whose equation is

$$3x - 4y - 8 + (y - 3z + 2) = 0, \text{ by (1)}$$

or, $$x - y - z = 2.$$

Note. In general, $\frac{\lambda}{\mu}$ will have two values, giving two tangent planes.

Examples V (B)

1. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 5$ at the point $(2, 0, 1)$.

2. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 4x - 2y + 6z - 32 = 0$ at the point $(1, 2, 3)$.

3. Find the equation of the tangent plane to the sphere $(x - 3)^2 + (y - 1)^2 + (z + 2)^2 = 24$ at the point $(-1, 3, 0)$.

4. Prove that the plane $2x + y - z = 12$ touches the sphere $x^2 + y^2 + z^2 = 24$ and find its point of contact.

5. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact.

6. (a) Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 = 9$ which are parallel to the plane

$$x + 2y - 2z + 15 = 0;$$

(b) Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 2 = 0$ parallel to the plane

$$x - y - z = 0.$$

[K. H. 2007]

(c) Find the equations of the tangent planes to the sphere $(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 25$ which are parallel to the plane

$$4x + 3z = 17.$$

7. Find the points on the sphere $x^2 + y^2 + z^2 - 2x + 4y - 7 = 0$ the tangent planes at which are parallel to the plane $x - y + z = 1$.

8. (a) Find the equations of the tangent planes to the sphere

$$x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$$

which pass through the straight line $3(16 - x) = 2y + 30 = 3z$.

(b) Show that the equations of the tangent planes to the sphere

$$x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$$

which intersect in the straight line $6x - 3y - 23 = 0 = 3z + 2$ are $2x - y + 4z - 5 = 0$ and $4x - 2y - z - 16 = 0$.

9. Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 - 10x + 2y + 26z - 113 = 0$ which are parallel to the straight lines $\frac{x+5}{2} = \frac{y-1}{-3} = \frac{z+13}{2}$ and $\frac{x+7}{3} = \frac{y+1}{-2} = \frac{z-8}{8}$.

10. (a) Find the equation of the sphere with the centre at the point $(1, -1, 3)$ and touching the plane $2x + y - 3z = 5$.

(b) Find the equation of the sphere which passes through the points $(1, 0, 0)$, $(2, -3, 4)$, $(4, 1, 0)$ and touches the plane

$$2x + 2y - z = 11.$$

11. (a) Show that the spheres $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch each other internally and find their point of contact.

(b) Show that the spheres $x^2 + y^2 + z^2 = 100$ and $x^2 + y^2 + z^2 - 24x - 30y - 32z + 400 = 0$ touch each other externally and find the point of contact.

12. (a) Determine the values of h for which the plane $x + y + z = h$ touches the sphere $x^2 + y^2 + z^2 = 48$.

(b) Find the values of a for which the plane $x + y + z = a\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

5.17. Angle of intersection of two non-concentric spheres.

The angle of intersection of two spheres is the angle between the tangent planes to the spheres at the common point. Since the radii of the spheres to the common point are normal to the corresponding normal planes, the angle between the spheres is the same as the angle between the radii drawn to the common point.

Let r_1 and r_2 be the radii of the two spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2g_1x + 2f_1y + 2h_1z + d_1 = 0,$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2g_2x + 2f_2y + 2h_2z + d_2 = 0$$

drawn from the centres C_1 and C_2 respectively to the common point P and let the distance $C_1 C_2 = d$. Let θ be the angle between the radii at P so that in the triangle C_1PC_2,

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}.$$

The two spheres will intersect only when θ is real,

that is, $-1 \le \cos\theta \le 1,$

that is, $-2r_1 r_2 \le r_1^2 + r_2^2 - d^2 \le 2r_1 r_2,$

that is, $r_1^2 + r_2^2 + 2r_1 r_2 \ge d^2$, which implies $r_1 + r_2 \ge d$

and $r_1^2 + r_2^2 - 2r_1 r_2 \le d^2$, which implies $|r_1 - r_2| \le d.$

If $r_1 + r_2 = d$, then the spheres touch each other externally. On the other hand, if $|r_1 - r_2| = d$, then the spheres touch each other internally.

In particular, if $\theta = \frac{1}{2}\pi$ then the spheres $S_1 = 0$ and $S_2 = 0$ will intersect each other *orthogonally*.

Thus, if $\theta = \frac{1}{2}\pi$, then $\cos\theta = 0$ and $r_1^2 + r_2^2 = d^2.$

Now $r_1^2 = g_1^2 + f_1^2 + h_1^2 - d_1,$

$r_2^2 = g_2^2 + f_2^2 + h_2^2 - d_2$

and $d^2 = (C_1 C_2)^2 = (-g_1 + g_2)^2 + (-f_1 + f_2)^2 + (-h_1 + h_2)^2.$

Hence the condition for orthogonal intersection of the spheres $S_1 = 0$ and $S_2 = 0$ is $g_1^2 + f_1^2 + h_1^2 - d_1 + g_2^2 + f_2^2 + h_2^2 - d_2$

$$= (-g_1 + g_2)^2 + (-f_1 + f_2)^2 + (-h_1 + h_2)^2$$

or, $2g_1g_2 + 2f_1f_2 + 2h_1h_2 = d_1 + d_2.$

5.18. Radical plane and radical line.

Let the equations of two non-concentric spheres $S_1 = 0$ and $S_2 = 0$ be given by

$$S_1 \equiv x^2 + y^2 + z^2 + 2g_1x + 2f_1y + 2h_1z + d_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + z^2 + 2g_2x + 2f_2y + 2h_2z + d_2 = 0.$$

Let $P(\alpha, \beta, \gamma)$ be such a point that the power of P with respect to the spheres $S_1 = 0$ and $S_2 = 0$ are equal.

Thus $\alpha^2 + \beta^2 + \gamma^2 + 2g_1\alpha + 2f_1\beta + 2h_1\gamma + d_1$

$$= \alpha^2 + \beta^2 + \gamma^2 + 2g_2\alpha + 2f_2\beta + 2h_2\gamma + d_2,$$

or, $2(g_1 - g_2)\alpha + 2(f_1 - f_2)\beta + 2(h_1 - h_2)\gamma + d_1 - d_2 = 0.$

Hence the locus of P is the plane

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + 2(h_1 - h_2)z + d_1 - d_2 = 0.$$

Thus the locus of a point in space such that the powers of the point with respect to two spheres are same is a plane. This plane is called the *radical plane* of the two spheres.

The radical planes of three spheres taken two at a time intersect in a straight line which is called the *radical line* of the three spheres.

The *radical centre* is defined as the point of intersection of four radical lines of four spheres taken three at a time.

It is clear that the direction ratios of the straight line joining the centres of the spheres $S_1 = 0$ and $S_2 = 0$ are

$$(g_1 - g_2),\ (f_1 - f_2),\ (h_1 - h_2).$$

But these are also the direction ratios of a normal to the radical plane. Hence the straight line joining the centres is perpendicular to the radical plane.

Furthermore, if the two spheres touch each other externally or internally, then their radical plane is a common tangent plane to the two spheres.

For two intersecting spheres, the circle of intersection lies on the radical plane.

Note. In general, if the coefficient of x^2 in the equations $S_1 = 0$ and $S_2 = 0$ be a_1 and a_2 respectively, then the radical plane is given by the equation

$$a_2S_1 - a_1S_2 = 0.$$

5.19. Co-axial system of spheres.

A system of spheres such that any two of them have the same radical plane is called the *co-axial system of spheres.*

Let the equations of two spheres be

$$S_1 \equiv x^2 + y^2 + z^2 + 2g_1x + 2f_1y + 2h_1z + d_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + z^2 + 2g_2x + 2f_2y + 2h_2z + d_2 = 0.$$

Then $$\lambda S_1 + \mu S_2 = 0, \quad \left(\frac{\lambda}{\mu} \neq -1\right), \qquad \ldots \quad (1)$$

where λ and μ are variables, represents a system of co-axial spheres.

For, let $S_1 + \lambda_1 S_2 = 0$ and $S_1 + \lambda_2 S_2 = 0$ be any two members of the system (1), then their radical plane is

$$(1 + \lambda_2)(S_1 + \lambda_1 S_2) - (1 + \lambda_1)(S_1 + \lambda_2 S_2) = 0,$$

that is, $(\lambda_2 - \lambda_1)(S_1 - S_2) = 0,$

that is, $S_1 - S_2 = 0,$

which is independent of the members of the system chosen.

The centre of the sphere belonging to a co-axial family is called a *limiting point* of the family, if the sphere has a zero radius.

5.20. Illustrative Examples.

Ex. 1. *Find the angle of intersection of the sphere*

$$x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$$

with the sphere, the extremities of whose diameter are the points (1, 2, – 3) *and* (5, 0, 1).

Equation of a sphere, the extremities of whose diameter are the points (1, 2, – 3) and (5, 0, 1) is

$$(x - 1)(x - 5) + (y - 2)(y - 0) + (z + 3)(z - 1) = 0$$

or, $x^2 + y^2 + z^2 - 6x - 2y + 2z + 2 = 0.$

Its centre is C_1 (3, 1, – 1) and radius is $r_1 = 3$. Centre of the given sphere is C_2 (1, 2, 3) and its radius is $r_2 = 2$. Hence the distance between the centres of the spheres is $C_1C_2 = d = \sqrt{21}$.

If θ be the required angle of intersection, then

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = \frac{9 + 4 - 21}{2.3.2} = -\frac{2}{3}.$$

Therefore $\theta = \cos^{-1}\left(-\frac{2}{3}\right)$, that is, $\cos^{-1}\frac{2}{3}$.

Ex. 2. *Find the limiting points of the co-axial system of spheres given by the equations* $x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$

and $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$.

The equation of the co-axial system of spheres is

$$x^2 + y^2 + z^2 + 3x - 3y + 6 + \lambda(x^2 + y^2 + z^2 - 6y - 6z + 6) = 0$$

or, $$(x^2 + y^2 + z^2)(1 + \lambda) + 3x - (3 + 6\lambda)y - 6\lambda z + 6(1 + \lambda) = 0.$$

The centre of the sphere is at the point

$$\left\{-\frac{3}{2(1+\lambda)}, \frac{3+6\lambda}{2(1+\lambda)}, \frac{3\lambda}{1+\lambda}\right\}.$$

If this be the limiting point of the system, then λ will be so chosen that the radius of the sphere is zero. Thus

$$\frac{9}{4(1+\lambda)^2} + \frac{(3+6\lambda)^2}{4(1+\lambda)^2} + \frac{9\lambda^2}{(1+\lambda)^2} - 6 = 0$$

or, $$8\lambda^2 - 2\lambda - 1 = 0.$$

This gives $\lambda = \frac{1}{2}$ or $-\frac{1}{4}$.

Hence the corresponding limiting points are

$$(-1, 2, 1), (-2, 1, -1).$$

Examples V(C)

1. Find the angle of intersection of the two spheres

$$x^2 + y^2 + z^2 - 6x - 2y + 2z + 2 = 0$$

and. $$x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0.$$

2. Prove that the sphere which cuts the two spheres $S = 0$ and $S' = 0$ at right angles, also cuts the sphere $\lambda S + \mu S' = 0$ at right angles.

3. Show that the spheres

$$x^2 + y^2 + z^2 - 2x + y - 3z + 4 = 0$$

and $$x^2 + y^2 + z^2 - 5x - 6y + 2z - 5 = 0$$

cut each other orthogonally.

4. Show that the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ is

$$x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0.$$

5. Show that the equation of the sphere which cuts orthogonally each of the spheres

$x^2 + y^2 + z^2 = 14$, $x^2 + y^2 + z^2 + 2x = 1$, $x^2 + y^2 + z^2 + 4y = 4$ and $x^2 + y^2 + z^2 + 6z = 9$ is $x^2 + y^2 + z^2 + 13x + 5y + \frac{5}{3}z + 14 = 0$.

6. Show that the locus of a point, whose powers with respect to two given spheres are in a constant ratio ($\neq 1$), is a sphere co-axial with the two given spheres.

7. Show that the equations of any two spheres can be put in the form $x^2 + y^2 + z^2 + 2\lambda x + d = 0$, $x^2 + y^2 + z^2 + 2\mu x + d = 0$.

[Take the line of centres as the x-axis and the radical plane as the yz-plane.]

8. Show that the spheres, which cut two given spheres along a great circle, all pass through two fixed points.

9. (a) Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of their common circle is $\dfrac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

(b) If two spheres intersect orthogonally, then show that the centre of each is an outside point of the other. [*K. H. 1997*]

10. Find the radical line of the spheres

$$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0,$$
$$x^2 + y^2 + z^2 + 4x + 4z + 4 = 0,$$
$$x^2 + y^2 + z^2 + x + 6y - 4z - 2 = 0.$$

11. Find, if possible, the radical centre of the four spheres, whose equations are

$$x^2 + y^2 + z^2 = 1,$$
$$x^2 + y^2 + z^2 - 10x = 0,$$
$$x^2 + y^2 + z^2 + 6z + 4 = 0,$$
$$x^2 + y^2 + z^2 - 4y - 12 = 0.$$

12. Find the limiting points of the co-axial system of spheres given by the equations

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0$$
and
$$x^2 + y^2 + z^2 + 2x - 4y - 2z + 6 = 0.$$

Answers

1. $\cos^{-1}\dfrac{13}{30}$. 10. $\dfrac{x}{2} = \dfrac{y-1}{5} = \dfrac{z}{3}$. 11. $\left(\dfrac{1}{10}, -\dfrac{11}{4}, -\dfrac{5}{6}\right)$.

12. $(-2, 1, -1)$, $(-1, 2, 1)$.

6 QUADRIC SURFACES

6.1. Quadric surface.

A surface defined in space by any equation of the second degree in x, y, z is called a *quadric surface* or simply a *quadric.* It is also known as a *conicoid.* The main characteristic property of such a surface is that a straight line cuts it in two points.

A surface of revolution is a surface generated by revolving a plane curve or a straight line in the plane of the curve. The plane curve is called the *generatrix* and the given line is called the *axis* of revolution.

A sphere is a surface which is generated by revolving a circle about a straight line which is the diameter of the circle.

The most general equation of the second degree in x, y, z,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

contains nine constants $\frac{b}{a}, \frac{c}{a}, \frac{f}{a}, \frac{g}{a}, \frac{h}{a}, \frac{u}{a}, \frac{v}{a}, \frac{w}{a}, \frac{d}{a}$, $(a \neq 0)$. For different values of these constants, this equation represents different types of surfaces, such as cone, cylinder, ellipsoid, hyperboloid, paraboloid, etc. Sphere is a special type of ellipsoid.

We shall study these surfaces one by one.

A. The Cone

6.2. Definitions.

A *cone* is a surface generated by a straight line which always passes through a fixed point called the *vertex* and intersects a given curve called the *guiding curve*; the moving straight line is called the *generator* of the cone. If the cone be such that its generator makes a constant angle with a fixed straight line through the vertex, then it is called *a right circular cone.* The fixed straight line is the *axis* of the cone and the constant angle is its *semi-vertical angle.*

6.3. Cone with its vertex at the origin.

Let us assume that the general equation of second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad \ldots (1)$$

represents a cone passing through the origin O.

If $P(x_1, y_1, z_1)$ be a point on the cone, then every point on OP, the generator of the cone, which is

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1},$$

must lie on the cone (1). Thus, for all values of r, (rx_1, ry_1, rz_1) must satisfy (1). Hence

$$r^2(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) + 2r(ux_1 + vy_1 + wz_1) + d = 0$$

must be an identity, so that

$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0, \qquad \ldots (2)$$

$$ux_1 + vy_1 + wz_1 = 0 \qquad \ldots (3)$$

and

$$d = 0. \qquad \ldots (4)$$

Now, if u, v, w be not all zero, then (3) shows that the co-ordinates of the point $P(x_1, y_1, z_1)$ which lies on a cone satisfy an equation of the first degree showing that the surface is a plane. But this is not possible. Hence $u = v = w = 0$ and by (4), $d = 0$.

Thus the equation of the cone becomes a *homogeneous* equation of the second degree, which is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad \ldots (5)$$

where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0.$$

Conversely, *every homogeneous equation of the second degree represents a cone with its vertex at the origin.*

If the homogeneous equation of the second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \quad \Delta \neq 0$$

be satisfied by any point $P(x_1, y_1, z_1)$, then for all values of r, the point (rx_1, ry_1, rz_1) also satisfies the above equation. But these are the general co-ordinates of a point on the straight line OP, O being the origin. Then every point on OP lies on the above surface. Hence the straight line OP lies entirely on it. Thus the surface is generated by straight lines through the origin and as such it is a cone whose vertex is at the origin.

Cor. If l, m, n be the direction ratios of the generator OP passing through the origin O, then the equations of OP are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

If any point (x', y', z') on OP satisfies the equation of the cone $f(x, y, z) = 0$, then obviously $f(l, m, n) = 0$, since l, m, n also will satisfy the equation.

On the other hand, if the direction cosines of a variable straight line passing through the origin be connected by the homogeneous relation $f(l, m, n) = 0$, then it generates a cone with vertex at the origin and whose equation is $f(x, y, z) = 0$.

Note. The equation (5) of the cone contains five arbitrary constants and hence it can be made to satisfy five conditions ; for example, the cone can be made to pass through any five concurrent lines.

6.4. Condition for the general equation of the second degree to represent a cone.

The most general equation of the second degree in x, y, z is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

Let us assume that this equation represents a cone with its vertex at the point (α, β, γ). Shifting the origin to the point (α, β, γ), the above equation transforms to

$$a(x+\alpha)^2 + b(y+\beta)^2 + c(z+\gamma)^2 + 2f(y+\beta)(z+\gamma)$$
$$+ 2g(z+\gamma)(x+\alpha) + 2h(x+\alpha)(y+\beta)$$
$$+ 2u(x+\alpha) + 2v(y+\beta) + 2w(z+\gamma) + d = 0.$$

This, when simplified, becomes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2x(a\alpha + h\beta + g\gamma + u)$$
$$+ 2y(h\alpha + b\beta + f\gamma + v) + 2z(g\alpha + f\beta + c\gamma + w)$$
$$+ a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta$$
$$+ 2u\alpha + 2v\beta + 2w\gamma + d = 0.$$

Now this equation represents a cone with its vertex at the origin. Hence this should be a homogeneous second degree equation in x, y and z.

Therefore $a\alpha + h\beta + g\gamma + u = 0,$ ... (1)

$h\alpha + b\beta + f\gamma + v = 0,$ ... (2)

$g\alpha + f\beta + c\gamma + w = 0$ ... (3)

and $a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 2v\beta + 2w\gamma + d = 0$;

that is, $\alpha(a\alpha + h\beta + g\gamma + u) + \beta(h\alpha + b\beta + f\gamma + v)$

$$+ \gamma(g\alpha + f\beta + c\gamma + w) + u\alpha + v\beta + w\gamma + d = 0,$$

that is,
$$u\alpha + v\beta + w\gamma + d = 0, \qquad \ldots \quad (4)$$

by (1), (2), (3).

Eliminating α, β, γ from the equations (1), (2), (3) and (4), we get the necessary condition for the general equation of the second degree to represent a cone as

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0.$$

When this condition holds good, the co-ordinates of the vertex (α, β, γ) are obtained by solving any three of the equations (1) to (4).

6.5. General equation of a cone containing the axes.

The general equation of the cone with the vertex at the origin is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, where $\Delta \neq 0$.

The straight line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on it, if

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0.$$

If it passes through the axes whose direction cosines are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, then these should satisfy the equation. Hence $a = b = c = 0$.

Thus the equation of the cone through the axes is

$$fyz + gzx + hxy = 0. \qquad \ldots \quad (1)$$

It can now easily be shown that the cone (1) can be made to pass through another set of rectangular axes through the same origin.

Let the direction ratios of the second set of axes be

$$l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3.$$

If the cone (1) contains the first two of these straight lines, then

$$fm_1n_1 + gn_1l_1 + hl_1m_1 = 0 \qquad \ldots \quad (2)$$

and

$$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0. \qquad \ldots \quad (3)$$

Now, since the new set of axes are rectangular, we have three relations of the type

$$m_1n_1 + m_2n_2 + m_3n_3 = 0.$$

Adding (2) and (3) and making use of these three relations, we get $fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$, which shows that the cone (1) also contains the third line of the second set of axes.

6.6. Equation of the cone with the origin as vertex and a given curve as base.

Let the guiding curve be $f(x, y) = 0,\ z = c$.

Let the straight line through the origin be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \qquad \ldots (1)$$

This meets the plane $z = c$, where $\frac{x}{l} = \frac{y}{m} = \frac{c}{n}$.

The point on the plane $z = c$, is thus $\left(\frac{lc}{n}, \frac{mc}{n}, c\right)$.

This lies on the curve $f(x, y) = 0,\ z = c$.

Therefore $\quad f\left(\frac{lc}{n}, \frac{mc}{n}\right) = 0.$

Eliminating l, m, n by (1), we have

$$f\left(\frac{xc}{z}, \frac{yc}{z}\right) = 0,$$

which is the equation of the cone, with the vertex at the origin.

If, in general, the vertex of the cone be at the point (α, β, γ), then the straight line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \qquad \ldots (2)$$

This meets the plane $z = c$, where $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{c-\gamma}{n}$.

The point on the plane $z = c$ is thus

$$\left\{\alpha + \frac{l(c-\gamma)}{n}, \beta + \frac{m(c-\gamma)}{n}, c\right\}.$$

This point lies on the curve $f(x, y) = 0,\ z = c$.

Therefore $f\left\{\alpha + \frac{l(c-\gamma)}{n}, \beta + \frac{m(c-\gamma)}{n}\right\} = 0.$

Eliminating l, m, n by (2), we get the equation of the cone as

$$f\left\{\alpha + \frac{(x-\alpha)(c-\gamma)}{z-\gamma}, \beta + \frac{(y-\beta)(c-\gamma)}{z-\gamma}\right\} = 0.$$

5.7. Equation of a right circular cone.

In a right circular cone, the generator makes a constant angle with the axis of the cone.

(i) Origin is at the vertex.

Let the axis of the cone be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and the semi-vertical angle be θ.

Let a, b, c be the direction ratios of a generator of the cone. So the equations of this generator are $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. ... (1)

Then this generator makes an angle θ with the axis whose direction ratios are l, m, n.

Therefore $\cos\theta = \dfrac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\,.\sqrt{a^2 + b^2 + c^2}}$

or, $(a^2 + b^2 + c^2)(l^2 + m^2 + n^2)\cos^2\theta = (al + bm + cn)^2$.

This is a homogeneous equation of second degree in a, b, c, which are the direction ratios of the generator of the cone.

Eliminating a, b, c from (1), we get the equation of the cone as

$$(l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\theta = (lx + my + nz)^2.$$

(ii) Vertex is at a point other than the origin.

Let (α, β, γ) be the vertex, θ be the semi-vertical angle and the axis of the cone be the straight line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

Let the straight line $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ ... (1)

be a generator of the cone. Then, θ being the constant angle between the axis and the generator, we have

$$\cos\theta = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\,.\sqrt{a^2 + b^2 + c^2}}$$

or, $(al + bm + cn)^2 = (l^2 + m^2 + n^2)(a^2 + b^2 + c^2)\cos^2\theta$. ... (2)

Eliminating a, b, c between (1) and (2), we get the equation of the cone as

$$\{l(x-\alpha) + m(y-\beta) + n(z-\gamma)\}^2$$
$$= (l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}\cos^2\theta.$$

Cor. If l, m, n be the direction cosines of a generator of the cone with vertex at the origin, z-axis as the axis and θ as the semi-vertical angle, then $n = \cos\theta$.

Therefore $\quad n^2 = \cos^2\theta\,(l^2 + m^2 + n^2)$

or, $\quad l^2 + m^2 = n^2 \tan^2\theta$.

Hence the equation of the cone is $x^2 + y^2 = z^2 \tan^2\theta$.

6.8. Intersection of a cone by a plane through the vertex.

Let the equation of a cone having the vertex at the origin be

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \ \Delta \neq 0 \quad \ldots (1)$$

and the plane be $\quad ux + vy + wz = 0. \quad \ldots (2)$

The lines of section are the two generators.

Let one of the lines in which the plane cuts the cone be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Since this line lies on (2) as well as on (1), we have

$$ul + vm + wn = 0 \quad \ldots (3)$$

and $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0. \quad \ldots (4)$

Eliminating n from (3) and (4), we have

$$al^2 + bm^2 + c\left(\frac{ul + vm}{w}\right)^2 - 2(fm + gl)\cdot\frac{ul + vm}{w} + 2hlm = 0$$

or, $l^2(cu^2 + aw^2 - 2gwu) + 2lm(hw^2 + cuv - fwu - gvw)$

$$+ m^2(bw^2 + cv^2 - 2fvw) = 0. \quad \ldots (5)$$

Dividing both sides by m^2, this becomes a quadratic equation in $\frac{l}{m}$ and this will give two values of this ratio corresponding to the two lines of intersection.

If, now (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction ratios of these two lines, then the equation (5) must be identical with

$$\left(\frac{l}{m} - \frac{l_1}{m_1}\right)\left(\frac{l}{m} - \frac{l_2}{m_2}\right) = 0,$$

that is, $\quad l^2 m_1 m_2 - lm(l_1 m_2 + l_2 m_1) + m^2 l_1 l_2 = 0. \quad \ldots (6)$

Comparing (5) and (6), we have

$$\frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{cu^2 + aw^2 - 2gwu}$$

$$= \frac{l_1 m_2 + l_2 m_1}{-2(hw^2 + cuv - fwu - gvw)} = k \text{ (say)}.$$

Now $(l_1 m_2 - l_2 m_1)^2 = (l_1 m_2 + l_2 m_1)^2 - 4 l_1 l_2 m_1 m_2$

$$= 4k^2 (hw^2 + cuv - fwu - gvw)^2$$

$$- 4k^2 (bw^2 + cv^2 - 2fvw)(cu^2 + aw^2 - 2gwu)$$

$$= 4k^2 \{h^2 w^4 - 2hw^3 (fu + gv) + w^2 (fu + gv)^2$$

$$+ 2chuvw^2 - 2cuvw (fu + gv) + c^2 u^2 v^2\}$$

$$- 4k^2 \{abw^4 - 2w^3 (afv + bgu) + w^2 (bcu^2 + cav^2)$$

$$- 2cuvw (fu + gv) + c^2 u^2 v^2 + 4fguvw^2\}$$

$= 4w^2 k^2 \{-(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)\}$

$= 4w^2 k^2 P^2,$

where A, B, C, F, G, H are the respective co-factors of a, b, c, f, g, h in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

and $P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$

By symmetry, we can now write the values of

$$n_1 n_2, \ (m_1 n_2 - m_2 n_1) \text{ and } (n_1 l_2 - n_2 l_1).$$

Thus $l_1 l_2 + m_1 m_2 + n_1 n_2$

$= k\{(bw^2 + cv^2 - 2fvw) + (cu^2 + aw^2 - 2gwu)$

$+ (av^2 + bu^2 - 2huv)\}$

$= k\{a(v^2 + w^2) + b(w^2 + u^2) + c(u^2 + v^2) - 2fvw - 2gwu - 2huv\}$

$= k\{(a + b + c)(u^2 + v^2 + w^2) - F(u, v, w)\}.$

Furthermore, $\Sigma (m_1 n_2 - m_2 n_1)^2 = 4k^2 P^2 (u^2 + v^2 + w^2).$

Hence, if θ be the angle between the lines of intersection, then

$$\frac{\cos\theta}{l_1l_2+m_1m_2+n_1n_2}=\frac{\sin\theta}{\left\{\Sigma(m_1n_2-m_2n_1)^2\right\}^{\frac{1}{2}}}$$

or,
$$\frac{\cos\theta}{(a+b+c)(u^2+v^2+w^2)-F(u,v,w)}=\frac{\sin\theta}{2P(u^2+v^2+w^2)^{\frac{1}{2}}}.$$

Cor. 1. The plane (2) cuts the cone (1) in perpendicular generators, if $\cos\theta=0$, that is, if $(a+b+c)(u^2+v^2+w^2)-F(u,v,w)=0$.

Cor. 2. The plane cuts the cone in two coincident generators, if $\sin\theta=0$, that is, if $P=0$,

that is, if $Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0$.

Note. If the angle between the two generators be zero, then the two generators coincide and the plane becomes a tangent plane to the cone.

6.9. Condition for three mutually perpendicular generators.

Let the equation of the cone be

$$F(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,\ \Delta\neq 0. \quad \ldots (1)$$

If the cone has three mutually perpendicular generators, we can choose these as the axes of co-ordinates. Then the equation takes the form

$$Lyx+Mzx+Nxy=0. \quad [\text{ Cf. Art. 6.5 }] \quad \ldots (2)$$

Since the sum of the coefficients of x^2, y^2 and z^2 is an invariant with respect to change of axes and that is zero in (2), hence the same quantity in (1) must be zero. Thus the necessary condition for a cone to have three mutually perpendicular generators is

$$a+b+c=0.$$

Conversely, if $a+b+c=0$, then there are infinite number of sets of three mutually perpendicular generators.

If we take any generator as the axis of x, then any point on the straight line $y=0$, $z=0$ is on the surface. Thus the coefficient of x^2 is zero and the transformed equation is of the form

$$b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0 \quad \ldots (3)$$

and by the invariants, $b'+c'=a+b+c=0$.

Now the section of (3) by the plane $x = 0$ is

$$b'y^2 + c'z^2 + 2f'yz = 0,$$

which gives two straight lines, perpendicular to one another, since

$$b' + c' = 0.$$

Moreover, these two perpendicular generators are each perpendicular to the new x-axis.

Thus the condition $a + b + c = 0$ ensures the sufficiency of the existence of three perpendicular generators.

Since the new axis of x can be chosen arbitrarily, the cone will have infinite sets of mutually perpendicular generators.

Note. The condition $a + b + c = 0$ may also be obtained in the following way :

The condition that the plane $ux + vy + wz = 0$ should cut the cone $F(x, y, z) = 0$ in perpendicular generators is

$$(a + b + c)(u^2 + v^2 + w^2) = F(u, v, w). \quad \ldots \quad \text{(A)}$$

Now, if the normal to the plane lies on the cone, we have

$$F(u, v, w) = 0. \quad \ldots \quad \text{(B)}$$

The condition (A), by virtue of (B), becomes $a + b + c = 0$.

In this case, the cone has three mutually perpendicular generators, namely, the normal to the plane and the two perpendicular lines in which the plane cuts the cone.

6.10. Illustrative Examples.

* Ex. 1. *Find the equation of the right circular cone which contains three positive co-ordinate axes.*

Here the axis of the cone makes equal angles with the co-ordinate axes. Therefore the semi-vertical angle of the cone is

$$\cos^{-1}\frac{1}{\sqrt{3}}.$$

The equations of the axis are $x = y = z$.

If (x, y, z) be any point on the cone, then the respective direction ratios of the generator through the point (x, y, z) and the axis are x, y, z and $1, 1, 1$.

The angle between them is $\cos^{-1}\frac{1}{\sqrt{3}}$.

Therefore $\frac{1}{\sqrt{3}} = \frac{x + y + z}{\sqrt{x^2 + y^2 + z^2}\,.\sqrt{1^2 + 1^2 + 1^2}}.$

Hence $x^2+y^2+z^2=(x+y+z)^2$, that is, $xy+yz+zx=0$.

This is the equation of the cone.

Ex. 2. *Find the equation of the cone whose vertex is the origin and base is the circle $x=a$, $y^2+z^2=b^2$.*

Show that the section of the cone by a plane parallel to XOY plane is a hyperbola.

The equation $y^2+z^2=b^2$ is made homogeneous with the help of the equation of the plane $x=a$, that is, $\frac{x}{a}=1$.

The result is $y^2+z^2=b^2\left(\frac{x}{a}\right)^2$

or, $$a^2(y^2+z^2)=b^2x^2.$$

This is the required equation of the cone.

Putting $z=c$, we get the section of the plane parallel to XOY plane as
$$b^2x^2-a^2y^2=a^2c^2,$$
which is a hyperbola.

Ex. 3. *Find the equation of the cone whose vertex is the point (1, 2, 3) and guiding curve is the circle $x^2+y^2+z^2=9$, $x+y+z=1$.*

Any generator of the cone through the point (1, 2, 3) is
$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{x+y+z-6}{l+m+n}. \quad \ldots (1)$$

It meets the plane $x+y+z=1$.

Therefore $$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{-5}{l+m+n}.$$

Hence $$x=1-\frac{5l}{l+m+n}=\frac{m+n-4l}{l+m+n},$$
$$y=2-\frac{5m}{l+m+n}=\frac{2l+2n-3m}{l+m+n},$$
$$z=3-\frac{5n}{l+m+n}=\frac{3l+3m-2n}{l+m+n}.$$

If this point lies on the sphere $x^2+y^2+z^2=9$, then
$$(m+n-4l)^2+(2l+2n-3m)^2+(3l+3m-2n)^2 = 9(l+m+n)^2. \quad \ldots (2)$$

Eliminating l, m, n between (1) and (2), we get the required equation of the cone as
$$(y+z-4x-1)^2+(2z+2x-3y-2)^2+(3x+3y-2z-3)^2=9(x+y+z-6)^2.$$

Simplifying, we get the result.

Ex. 4. Find the equation of the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface

$$ax^2 + by^2 + cz^2 = 1.$$

We know that a homogeneous equation of the second degree in x, y, z represents a cone with vertex at the origin.

From the equation of the plane $lx + my + nz = p$, we have

$$\frac{lx + my + nz}{p} = 1. \quad \ldots \quad (1)$$

Now making $ax^2 + by^2 + cz^2 = 1$ homogeneous in x, y, z of degree two by (1), we get

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2.$$

This homogeneous equation represents a cone, because it is satisfied by the co-ordinates of any point which satisfy the equations of the plane and the surface simultaneously.

This is the required equation of the cone.

Ex. 5. *Find the equations of the straight lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Find also the angle between them.*

Eliminating z between the two equations, we get

$$4x^2 - y^2 + 3(2x + y)^2 = 0$$

or,
$$16x^2 + 12xy + 2y^2 = 0.$$

Solving, $\frac{x}{y} = -\frac{1}{2}$ or $-\frac{1}{4}$.

From the equation of the plane by putting these values for $\frac{x}{y}$, we have

$$\frac{z}{y} = 0 \text{ or } \frac{1}{2}.$$

Thus we have two sets of values of the ratio $x : y : z$ given by

$$x : y : z = -1 : 2 : 0 \text{ or } -1 : 4 : 2.$$

Thus the equations of the two lines of section are

$$\frac{x}{-1} = \frac{y}{2} = \frac{z}{0} \text{ and } \frac{x}{-1} = \frac{y}{4} = \frac{z}{2}.$$

If θ be the angle between these two straight lines, then

$$\cos\theta = \frac{1 + 8 + 0}{\sqrt{5}.\sqrt{21}}.$$

Therefore $\theta = \cos^{-1}\frac{9}{\sqrt{105}}.$

Ex. 6. *(a) The plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *meets the co-ordinate axes at A, B, C. Find the equation of the cone generated by the straight lines drawn from O to meet the circle ABC.* [B. H. 1986, 2003]

(b) Find the locus of points from which three mutually perpendicular straight lines can be drawn to intersect a given circle.

(*a*) The plane meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

The equation of the sphere through O, A, B, C is

$$x^2+y^2+z^2-ax-by-cz=0.$$

Therefore the equations of the circle are given by the equations

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots (1)$$

and $$x^2+y^2+z^2-ax-by-cz=0. \qquad \ldots (2)$$

The required equation is obtained by making (2) homogeneous of degree two in x, y, z with the help of (1) and is thus

$$x^2+y^2+z^2=(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)$$

or, $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

(*b*) Let the equation of the given circle be $x^2+y^2=a^2$, $z=0$.

The straight line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ meets the plane $z=0$ at the point

$$\left(\frac{\alpha n-\gamma l}{n}, \frac{\beta n-\gamma m}{n}, 0\right).$$

If this point lies on the given circle, then

$$\frac{(\alpha n-\gamma l)^2}{n^2}+\frac{(\beta n-\gamma m)^2}{n^2}=a^2.$$

Hence the locus of the lines drawn from the point (α, β, γ) to intersect the given circle is the cone whose equation is

$$\{\alpha(z-\gamma)-\gamma(x-\alpha)\}^2+\{\beta(z-\gamma)-\gamma(y-\beta)\}^2=a^2(z-\gamma)^2$$

or, $$(\alpha z-\gamma x)^2+(\beta z-\gamma y)^2=a^2(z-\gamma)^2.$$

This cone will have three mutually perpendicular generators, if

$$(\alpha^2+\gamma^2)+(\beta^2+\gamma^2)=a^2.$$

Hence the locus of the point (α, β, γ) from which three mutually perpendicular lines can be drawn to intersect the given circle is

$$x^2+y^2+2z^2=a^2.$$

Ex. 7. *If the straight line* $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ *represents one of a set of three mutually perpendicular generators of the cone* $5yz-8zx-3xy=0$, *then find the equations of the other two.*

[B. H. 1990; N. B. H. 1993; C. H. 1998]

Therefore $\lambda = \frac{l}{m} = \frac{p}{q}$. ... (1)

Again OP describes the cone $f\left(\frac{y}{x}, \frac{z}{x}\right) = 0$ and hence the direction ratios of OP will satisfy this equation. Thus

$$f\left(\frac{m}{l}, \frac{n}{l}\right) = 0. \quad \ldots (2)$$

OP and OQ being perpendicular to each other,

$$pl + qm + rn = 0.$$

Therefore $\quad p + q \cdot \frac{q}{p} + r \cdot \frac{n}{l} = 0$, from (1).

Hence $\quad \frac{n}{l} = -\frac{p}{r} - \frac{q^2}{pr}$. (3)

Putting these values of $\frac{m}{l}$ from (1) and $\frac{n}{l}$ from (3) in equation (2), we have

$$f\left(\frac{q}{p}, -\frac{p}{r} - \frac{q^2}{pr}\right) = 0.$$

Thus the line OQ with direction ratios p, q, r generates the cone

$$f\left\{\frac{y}{x}, \left(-\frac{x}{z} - \frac{y^2}{zx}\right)\right\} = 0.$$

Examples VI(A)

1. The axis of a right circular cone with vertex at the origin makes equal angles with the co-ordinate axes and the cone passes through the line drawn from the origin with direction cosines proportional to (1, –2, 2). Show that the equation of the cone is

$$4(x^2 + y^2 + z^2) + 9(xy + yz + zx) = 0.$$

2. (*a*) Find the equation of the right circular cone whose vertex is the origin, axis is the x-axis and semi-vertical angle is 60°.

(*b*) Find the equation of the right circular cone whose vertex is the origin, axis is the z-axis and which passes through the point (3, 4, – 6).

(*c*) Show that the equation of the right circular cone, whose vertex is the origin and semi-vertical angle is 45° and axis is $x = y = z$, is $x^2 + y^2 + z^2 = 4(xy + yz + zx)$.

3. Show that the semi-vertical angle of the right circular cone

$$4(x^2 + y^2) - 9z^2 = 0 \text{ is } \tan^{-1}\frac{3}{2}.$$

4. (a) Find the equation of the right circular cone whose vertex is the origin and whose

(i) axis is $\frac{x}{3}=\frac{y}{2}=\frac{z}{4}$ and semi-vertical angle is 45° ;

(ii) axis is $\frac{x}{2}=\frac{y}{-1}=\frac{z}{2}$ and semi-vertical angle is 60°

(b) Find the equation of the right circular cone which passes through point (1, 1, 2), has its vertex at the origin and has the line $\frac{x}{2}=\frac{y}{4}=\frac{z}{3}$ its axis. [B. H. 1997]

5. Find the equation of the right circular cone with

(i) vertex at the point (1, –2, –1), semi-vertical angle 60° and axis

$$\frac{x-1}{3}=\frac{y+2}{-4}=\frac{z+1}{5};$$

(ii) vertex at the point (3, 2, 1), semi-vertical angle 30° and axis

$$\frac{x-3}{1}=\frac{y-2}{4}=\frac{z-1}{3}.$$

6. (a) Show that the equation of the cone whose vertex is the origin and which passes through the curve of intersection of

(i) $4x^2-5y^2-7z^2=2,\ 3x-2y+4z=3$ is

$$9(4x^2-5y^2-7z^2)=2(3x-2y+4z)^2;$$

(ii) $Ax^2+By^2=1,\ z=C$ is $C^2(Ax^2+By^2)=z^2$;

(iii) $3x^2+6yz-6y+5z+7=0,\ 2x+y-3z+4=0$ is

$$16(3x^2+6yz)+4(6y-5z)(2x+y-3z)+7(2x+y-3z)^2=0.$$

(b) A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $\left(\frac{b}{c}+\frac{c}{b}\right)yz+\left(\frac{c}{a}+\frac{a}{c}\right)zx+\left(\frac{a}{b}+\frac{b}{a}\right)xy=0.$

[If the parallel plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$, then the equation of the sphere OABC is $x^2+y^2+z^2=k(ax+by+cz)$. Then eliminate k.]

(c) Find the equation of the cone whose vertex is at the origin and which contains the curve given by

$$x^2-y^2+4ax=0,\ x+y+z=b.$$

(d) Show that the equation of the cone whose vertex is (1, 0, –1) and which passes through the circle $x^2+y^2+z^2=4,\ x+y+z=1$ is $(x-1)^2+y^2+(z+1)^2+2(x-z-2)(x+y+z)=2(x+y+z)^2.$ [K. H. 1998]

Answers

2. (a) $3x^2 - y^2 + z^2$. (b) $36x^2 + 36y^2 - 25z^2$.

4. (a) (i) $29(x^2+y^2+z^2) = 2(3x+2y+4z)^2$.

(ii) $7(x^2+y^2+z^2) = 2(2x-y+3z)^2$.

(b) $24(x^2+y^2+z^2) = (2x+4y+3z)^2$.

5. (i) $2(3x-4y+5z-6)^2 = 25\{(x-1)^2+(y+2)^2+(z+1)^2\}$.

(ii) $2(x+4y+3z-14)^2 = 39\{(x-3)^2+(y-2)^2+(z-1)^2\}$.

6. (c) $b(x^2-y^2)+4ax(x+y+z)=0$.

7. $3x^2+2y^2+z^2-7xy-5yz+6zx=0$.

8. $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$.

10. (a) $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$; $\frac{x}{-1} = \frac{y}{1} = \frac{z}{2}$. (b) $\frac{x}{3} = \frac{y}{-2} = \frac{z}{-1}$; $\frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$.

(c) $\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$; $\frac{x}{15} = \frac{y}{10} = \frac{z}{-6}$.

11. (b) $9x+10y+8z=0$.

14. $\cos^{-1}\frac{5}{6}$.

15. (a) (i) $x^2+y^2=z^2$. (ii) $(z-x)^2+(z-y)^2 = 16(z-1)^2$.

(iii) $4x^2-4y^2 = 3(z-2)^2$. (iv) $2(3x-z)^2+3(3y-2z)^2 = (z-3)^2$.

(v) $5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y+10z-26=0$.

(vi) $a(\alpha z-\gamma x)^2 + 2h(\alpha z-\gamma x)(\beta z-\gamma y) + b(\beta z-\gamma y)^2 + 2g(\alpha z-\gamma x)(z-\gamma)$
$+ 2f(\beta z-\gamma y)(z-\gamma) + c(z-\gamma)^2 = 0$.

(c) $(3x-z)^2+(3y-2z)^2 = 25(z-3)^2$.

(d) $3(5y+5z-7x)^2+2(4z-3y)^2 = 6(y+z-7)^2$.

18. $(x-1)^2 = 4(y^2+z^2)$; $(1, 0, 0)$.

22. $\frac{x}{2} = \frac{y}{-3} = \frac{z}{4}$; $\frac{x}{-11} = \frac{y}{2} = \frac{z}{7}$.

B. The cylinder

6.11. Definitions.

A *cylinder* is a surface generated by the movement of a straight line which passes through a fixed curve and remains parallel to a fixed straight line. The fixed curve is called the *guiding curve*, the fixed straight line is called the *axis* and the moving straight line is called the *generator* of the cylinder.

A cylinder is also obtained when the moving line touches a given surface.

The cylinder may be regarded as a cone having the vertex at infinity.

If the plane curve be a circle and the axis be perpendicular to the circle at the centre, then the cylinder is called a *right circular cylinder.*

6.12. Equation of a cylinder.

Let the guiding curve of the cylinder be $f(x, y) = 0$, $z = 0$ and let the generator be parallel to the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Let (x', y', z') be a point on the cylinder, so that the equations of the generator are $\dfrac{x - x'}{l} = \dfrac{y - y'}{m} = \dfrac{z - z'}{n}$.

The point $(x' + lr, y' + mr, z' + nr)$ lies on the curve

$$f(x, y) = 0, z = 0.$$

Hence $z' + nr = 0$, that is, $r = -\dfrac{z'}{n}$.

Therefore $f(x' + lr, y' + mr) = 0 = f\left(x' - \dfrac{lz'}{n}, y' - \dfrac{mz'}{n}\right)$.

The equation of the cylinder is the locus of (x', y', z') and is thus

$$f\left(x - \frac{lz}{n}, y - \frac{mz}{n}\right) = 0.$$

In particular, if the given curve be $ax^2 + by^2 = 1$, $z = 0$, then the equation of the cylinder is

$$a\left(x - \frac{lz}{n}\right)^2 + b\left(y - \frac{mz}{n}\right)^2 = 1$$

or,

$$a(nx - lz)^2 + b(ny - mz)^2 = n^2.$$

6.13. Equation of a right circular cylinder.

Let the cylinder be of radius r and let the axis be the straight line ... (1)

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}.$$

Let (x', y', z') be any point on the cylinder. Then the square of the perpendicular distance of this point from the straight line (1) is

$$(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2-\left\{\frac{l(x'-\alpha)+m(y'-\beta)+n(z'-\gamma)}{\sqrt{l^2+m^2+n^2}}\right\}^2.$$

The locus of (x', y', z') will be the right circular cylinder, if this distance be equal to r, the radius of the cylinder.

Hence the equation of the right circular cylinder is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\left\{\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right\}^2=r^2.$$

Cor. The equation of the right circular cylinder whose axis is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, is

$$(ny-mz)^2+(lz-nx)^2+(mx-ly)^2=r^2(l^2+m^2+n^2).$$

6.14. Illustrative Examples.

Ex. 1. *Find the equation of the cylinder, whose generators are parallel to the straight line* $\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$ *and whose guiding curve is* $x^2+y^2=9$, $z=1$.

[K. U. 1986]

Let (x', y', z') be a point on the curve $x^2+y^2=9$, $z=1$. Now the equation of the straight line parallel to the line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$ passing through the point (x', y', z') is $\frac{x-x'}{-1}=\frac{y-y'}{2}=\frac{z-z'}{3}$.

This straight line cuts the curve $x^2+y^2=9$, $z=1$

at $\frac{x-x'}{-1}=\frac{1-z'}{3}$, or, $x=x'+\frac{z'-1}{3}$

and $\frac{y-y'}{2}=\frac{1-z'}{3}$, or, $y=y'-\frac{2(z'-1)}{3}$.

Substituting these for x, y in $x^2+y^2=9$ and making (x', y', z') current, we get the locus as

$$\left(x+\frac{z-1}{3}\right)^2+\left\{y-\frac{2(z-1)}{3}\right\}^2=9$$

or,

$$(3x+z-1)^2+(3y-2z+2)^2=81$$

or,

$$9x^2+9y^2+5z^2+6xz-12yz-6x+12y-10z-76=0.$$

Ex. 2. *Find the equation of the right circular cylinder of radius 3 and whose axis is* $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$.

The direction cosines of the axis are $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$.

The length of the perpendicular from a point (x, y, z) to the axis is

$(x-1)^2+(y-2)^2+(z-3)^2-\{\frac{2}{7}(x-1)-\frac{3}{7}(y-2)+\frac{6}{7}(z-3)\}^2$.

Hence the equation of the cylinder is

$(x-1)^2+(y-2)^2+(z-3)^2-\frac{1}{49}\{2(x-1)-3(y-2)+6(z-3)\}^2=9$.

Ex. 3. *Find the equation of the cylinder whose generating line is parallel to the z-axis and the guiding curve is*

$$x^2+y^2=z,\ x+y+z=1.$$

Let (α, β, γ) be a point on the curve.

Then $\alpha^2+\beta^2=\gamma$ and $\alpha+\beta+\gamma=1$. ... (1)

If (x, y, z) be any point on the cylinder, then the straight line joining the points (α, β, γ) and (x, y, z) is parallel to the z-axis whose direction cosines are 0, 0, 1. The equations are

$$\frac{x-\alpha}{0}=\frac{y-\beta}{0}=\frac{z-\gamma}{1}, \text{ that is, } x=\alpha,\ y=\beta.$$

Substituting these in (1), we get $x^2+y^2=\gamma,\ x+y+\gamma=1$.

Eliminating γ, we get the equation of the cylinder as

$$x^2+y^2+x+y=1.$$

Ex. 4. *Find the equation of the cylinder generated by straight lines parallel to the z-axis and passing through the curve of intersection of the plane*

$$lx+my+nz=p \quad \ldots (1)$$

and the surface

$$ax^2+by^2+cz^2=1. \quad \ldots (2)$$

Eliminating z between (1) and (2), we get the equation

$$ax^2+by^2+c\left(\frac{p-lx-my}{n}\right)^2=1. \quad \ldots (3)$$

The equation (3) represents a cylinder generated by straight lines parallel to the z-axis. Moreover, it is satisfied by the co-ordinates of a point which satisfy equations (1) and (2) together. Hence the cylinder given by (3) passes through the curve of intersection of (1) and (2). Thus (3) is the required equation of the cylinder.

Examples VI(B)

1. Show that the equation of the right circular cylinder of radius 2 whose axis passes through the point (1, 2, 3) and has direction ratios (2, −3, 6) is

$$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0.$$

2. (*a*) Show that the equation of the right circular cylinder, whose guiding curve is the circle through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is $x^2+y^2+z^2-yz-zx-xy=1$. [K. H. 2007]

(*b*) Show that the equation of the circular cylinder, whose guiding circle is $x^2+y^2+z^2=9$, $x-y+z=3$, is $x^2+y^2+z^2+xy+yz-zx=9$.

3. A cylinder has for its guiding curve the ellipse $4x^2+y^2=1$, $z=0$ and its generating line is parallel to the straight line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{3}$. Show that the equation of the cylinder is

$$36x^2+9y^2+17z^2+6yz-48zx-9=0.$$

4. Find the equation of the right circular cylinder, whose axis is

(*i*) $\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$ and radius equal to 2 ;

(*ii*) $\frac{x}{2}=\frac{y}{3}=\frac{z}{6}$ and radius equal to $\sqrt{5}$;

(*iii*) $\frac{x}{1}=\frac{y}{0}=\frac{z}{-2}$ and radius equal to 7 ;

(*iv*) *z*-axis and radius equal to 1 ;

(*v*) the straight line which passes through the point (1, 3, 4) and has 1, (–2), 3 as its direction ratios and radius equal to 3. [C. H. 1982]

5. Find the equation of the cylinder, whose generators are parallel to the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ and which passes through the conic

$$z=0,\ 3x^2+7y^2=12.$$

6. Find the equation of the cylinder generated by straight lines parallel to $\frac{x}{1}=\frac{y}{5}=\frac{z}{-2}$, the guiding curve being the conic $x=0$, $y^2=6z$.

7. Find the equation of the cylinder, whose generators are parallel to the straight line $-3x=6y=2z$ and whose guiding curve is the ellipse $2x^2+y^2=1$, $z=0$.

8. Find the equation of the cylinder, whose generators are parallel to the straight line $2x=y=3z$ and which passes through the circle

$$y=0,\ x^2+z^2=6.$$

9. Obtain the equation of the cylinder, whose generators intersect the plane curve $ax^2+by^2=1$, $z=0$ and are parallel to the straight line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

10. Obtain the equation of the cylinder, whose generators intersect the ellipse $9x^2 + 3y^2 = 1$, $z = 0$ and are parallel to the straight line with direction ratios $1, (-1), 1$. [K. H. 2003]

11. P is a variable point such that its distance from the xy-plane is always equal to one-fourth the square of its distance from the y-axis. Show that the locus of P is a cylinder. [B. H. 1990]

12. (a) Find the equation of the cylinder generated by straight lines parallel to the z-axis and passing through the curve of intersection of the plane $3x + 2y - z = 5$ and the surface $5x^2 - 2y^2 + 7z^2 = 1$.

(b) Find the equation of the cylinder, whose generators are parallel to the x-axis and which passes through the curve of intersection of the plane $2x - 3y + z = 2$ and the surface $3y^2 - 5z^2 = 10x$.

13. Find the equation of the cylinder, whose generators are parallel to the y-axis and which passes through the curve of intersection of the plane $x + y + z = 4$ and the surface $x^2 + y^2 + z^2 = 4$.

14. Show that the equation of the right circular cylinder, which passes through the point $(3, -1, 1)$ and has the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{1}$ as its axis, is

$$2x^2 + 5y^2 + 5z^2 + 4xy + 2yz - 4zx + 16x + 22y - 10z = 18.$$

15. Show that the equation of the cylinder, whose generators are parallel to the straight line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1$, $z = 3$, is

$$3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0.$$

16. Find the equation of the cylinder, whose generators are parallel to the straight line $\frac{x}{3} = \frac{y-4}{5} = \frac{z+1}{-4}$ and which has for its guiding curve the hyperbola $3x^2 - 4y^2 = 5$, $z = 2$.

Answers

4. (i) $8x^2 + 5y^2 + 5z^2 + 4xy + 8yz - 4zx = 36$.

(ii) $49(x^2 + y^2 + z^2 - 5) = (2x + 3y + 6z)^2$.

(iii) $5(x^2 + y^2 + z^2 - 49) = (x - 2z)^2$.

(iv) $x^2 + y^2 = 1$.

(v) $13x^2 + 10y^2 + 5z^2 + 4xy - 12yz - 6zx - 14x - 112y - 70z + 189 = 0$.

5. $3(5x - 2z)^2 + 7(5y - 3z)^2 = 300$.

6. $(y - 5x)^2 = 6(z + 2x)$.

7. $2(3x + 2z)^2 + (3y - z)^2 = 9$.

8. $9(2x - y)^2 + 4(3z - y)^2 = 216$.

9. $a(nx - lz)^2 + b(ny - mz)^2 = n^2$.

10. $9(x - z)^2 + 3(y + z)^2 = 1$.

12. (*a*) $5x^2 - 2y^2 + 7(3x + 2y - 5)^2 = 1$.
(*b*) $3y^2 - 5z^2 - 15y + 5z - 10 = 0$.

13. $x^2 + z^2 + xz - 4x - 4z + 6 = 0$.

16. $48x^2 - 64y^2 - 73z^2 - 160yz + 72zx - 144x + 320y + 292z - 372 = 0$.

C. Central conicoids

6.15. Ellipsoids.

Let us consider the surface given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \ldots \quad (1)$$

in which a, b, c are real numbers, none of them being zero. The surface given by (1) is called an *ellipsoid*, the equation being in its canonical form.

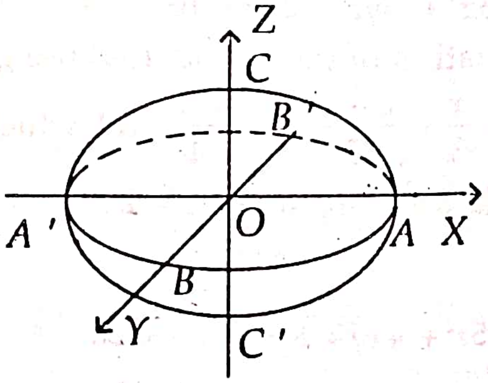

Fig. 19

(*i*) Let $P(\alpha, \beta, \gamma)$ be a point on the ellipsoid (1), so that

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1,$$

which can be put as $\frac{(-\alpha)^2}{a^2}+\frac{(-\beta)^2}{b^2}+\frac{(-\gamma)^2}{c^2}=1$,

which suggests that the point $P'(-\alpha, -\beta, -\gamma)$ also lies on (1). Thus the surface is *symmetric with respect to the origin*. Hence all chords of the ellipsoid passing through the origin, are bisected at the origin, which is thus the *centre* of the ellipsoid.

(*ii*) It is easy to verify that if the point $P(\alpha, \beta, \gamma)$ lies on (1), then the point $P'(\alpha, \beta, -\gamma)$ also lies on it. Now the middle point of PP' is $(\alpha, \beta, 0)$ which lies on the xy-plane, that is, the plane $z = 0$. Furthermore, the straight line joining P and P' is perpendicular to this plane. Thus the surface is symmetrical with respect to the xy-plane and similarly with respect to the yz-plane and zx-plane. These planes are called the *principal planes* of the ellipsoid and the intersection of the three principal planes taken in pairs are called the *principal axes* of the ellipsoid, which are the co-ordinate axes.

(*iii*) The surface is symmetrical with respect to the x-axis which is clear from the fact that the equation remains unchanged on changing (x, y, z) to $(x, -y, -z)$. Similarly, the surface is symmetrical with respect to y- and z-axes.

The z-axis $(x = 0, y = 0)$ meets the surface (1) in points $(0, 0, c)$ and $(0, 0, -c)$ such that the intercepts on the z-axis is $2c$. Similarly the intercepts on x- and y-axes are $2a$ and $2b$ respectively. These intercepts on the co-ordinate axes are called the *lengths of the axes*. $2a$ is the length of the major axis, $2b$ is the length of the mean axis and $2c$ is the length of the minor axis, if $a > b > c$. In this case, the ellipsoid is referred to as triaxial.

If any two of them be equal, then the ellipsoid is an *ellipsoid of revolution*. If $a = b$, then z-axis is the axis of revolution.

If $a = b < c$, then the ellipsoid is called a *prolate spheroid*

If $a = b > c$, then it is called an *oblate spheroid*.

If $a = b = c$, then the ellipsoid is a sphere.

(*iv*) If x be numerically greater than a, then from (1), we see that $\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$ is a negative quantity, so that either y^2 or z^2 must be negative, that is, either y or z is imaginary. Hence it is concluded that x cannot be numerically greater than a, or in other words, the surface lies between the two parallel planes $x = a$ and $x = -a$.

Similarly, it lies between the parallel planes

$$y = b,\ y = -b \text{ and } z = c,\ z = -c.$$

Thus *the ellipsoid is a closed surface.*

(*v*) The section of the ellipsoid (1) by the plane $z = k$ has for its equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2} = \frac{c^2 - k^2}{c^2}$, $z = k$.

This section is an ellipse, if $c^2 > k^2$, that is, $-c < k < c$ with semi-axes $a\sqrt{1 - \frac{k^2}{c^2}}$ and $b\sqrt{1 - \frac{k^2}{c^2}}$.

If $k > c$, then the section is not real; hence $k > c$ is an impossibility, since the ellipsoid is a closed surface.

If $k = c$ or $-c$, then the semi-axes of the ellipse are zero and we say that the section reduces to the point $(0, 0, c)$ or $(0, 0, -c)$.

If $k = 0$, then the section is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

For different values of k ranging from 0 to c, we obtain different ellipses having their centres lying upon the z-axis and their sizes continually diminish. Some condition occurs as k changes from 0 to $(-c)$.

Similarly, the section by planes parallel to the planes $x = k$ and $y = k$ are also ellipses. Hence *an ellipsoid is generated by a variable ellipse.*

6.16. Hyperboloids.

(*a*) *Hyperboloid of one sheet.*

Let us consider the surface given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad \ldots (1)$$

in which a, b, c are real, none being equal to zero. The surface given by (1) is called a *hyperboloid of one sheet*, the equation is said to be in canonical form.

(*i*) As in the case of an ellipsoid, the surface is symmetrical about the origin, which is the centre of the surface.

(*ii*) The co-ordinate planes bisect all chords perpendicular to them. These are then the principal planes. The surface is also symmetrical with respect to the co-ordinate axes, which are the principal axes of the surface.

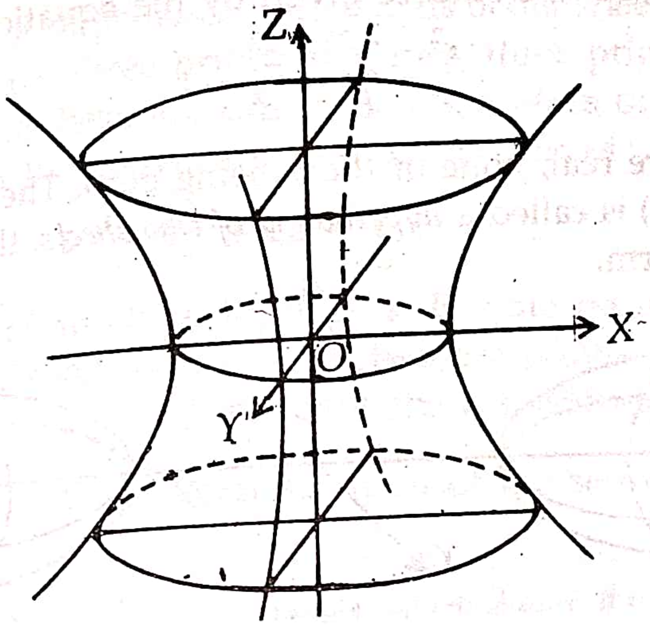

Fig. 20

(*iii*) The surface makes intercepts $2a$ and $2b$ on the co-ordinate axes of x and y whereas the axis of z does not meet the surface in real points.

(*iv*) The section by the plane $z = k$, which is parallel to the xy-plane, is an ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2},\ z = k,$$

whose centre lies on the z-axis and whose semi-axes are

$$a\sqrt{1 + \frac{k^2}{c^2}},\ b\sqrt{1 + \frac{k^2}{c^2}}.$$

As k increases, the size of the ellipse increases indefinitely, the centre remaining always on the z-axis. Similarly, the section by the plane $z = -k$ will behave in the same manner, being always on the other side of the xy-plane.

Again the section by the planes $x = k$ or $y = k$ parallel to yz- or zx-plane is each a hyperbola.

Note. The equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

also represent hyperboloids of one sheet, the axis of the former being the y-axis while the axis of the latter is the x-axis.

(b) *Hyperboloid of two sheets.*

Let us consider the surface given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \ldots \quad (1)$$

where a, b, c are real, none of them being zero. The surface given by the equation (1) is called a *hyperboloid of two sheets*, the equation being in canonical form.

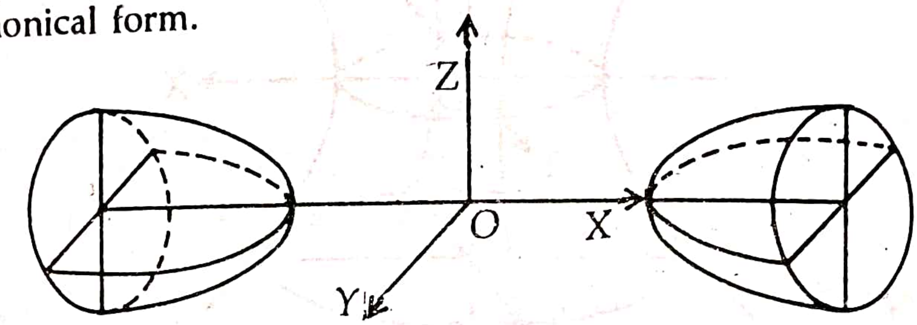

Fig. 21

(i) The surface is symmetrical about the origin, which is the centre of the surface.

(ii) The co-ordinate planes bisect all chords perpendicular to them so that the surface is symmetrical about the co-ordinate planes which are its principal planes and their lines of intersection taken in pairs are the principal axes.

(iii) The surface makes an intercept $2a$ on the x-axis while the y- and z-axes do not meet the surface in real points.

(iv) The section by the plane $x = k$, parallel to yz-plane is the ellipse given by

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1, \; x = k.$$

The centres of these ellipses lie on the x-axis. If k be numerically greater than a, then these ellipses will be real and they will increase in size with increase in k beyond a. If k be numerically less than a, then the ellipses are imaginary and hence the hyperboloid of two sheets will have no point between the planes $x = a$ and $x = -a$.

(v) The section by the plane $y = k$ or $z = k$ is a hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}, \; y = k \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \; z = k.$$

Note. The equations

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ and } -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

also represent hyperboloids of two sheets.

6.17. General equation of a central conicoid.

The surfaces of ellipsoids, hyperboloids of one sheet or hyperboloids of two sheets have three principal planes, three principal axes and a centre and hence these are called *central quadrics* or *central conicoids*. The equation of the above surfaces may be put as

$$ax^2 + by^2 + cz^2 = 1.$$

In the case of an ellipsoid, all of a, b, c are positive while in the case of a hyperboloid of one sheet, two are positive and in the case of a hyperboloid of two sheets, only one is positive.

Consider now the general equation of the second degree in the form

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0. \quad \ldots \quad (1)$$

This equation can be put in the form

$$a\left(x + \frac{u}{a}\right)^2 + b\left(y + \frac{v}{b}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} - d.$$

Putting $x + \frac{u}{a} = X$, $y + \frac{v}{b} = Y$, $z + \frac{w}{c} = Z$

and $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} - d = k$, we have

$$\frac{a}{k}X^2 + \frac{b}{k}Y^2 + \frac{c}{k}Z^2 = 1, \quad \ldots \quad (2)$$

which represents a central conicoid with principal axes

$$Y = 0 = Z,\ Z = 0 = X,\ X = 0 = Y.$$

Therefore the equation (1) represents a central conicoid, whose principal axes are

$$y + \frac{v}{b} = 0 = z + \frac{w}{c},\ z + \frac{w}{c} = 0 = x + \frac{u}{a},\ x + \frac{u}{a} = 0 = y + \frac{v}{b}.$$

Principal planes are $x + \frac{u}{a} = 0$, $y + \frac{v}{b} = 0$, $z + \frac{w}{c} = 0$.

The co-ordinates of the centre are $\left(-\frac{u}{a}, -\frac{v}{b}, -\frac{w}{c}\right)$.

6.18. Illustrative Examples.

Ex. 1. *Find the nature of the quadric surface given by the equation*

$$2x^2 + 5y^2 + 3z^2 - 4x + 20y - 6z = 5.$$

The given equation can be written as

$$2(x-1)^2 + 5(y+2)^2 + 3(z-1)^2 = 30$$

or,
$$\frac{(x-1)^2}{15} + \frac{(y+2)^2}{6} + \frac{(z-1)^2}{10} = 1.$$

Shifting the origin to the point $(1, -2, 1)$, the equation reduces to

$$\frac{X^2}{(\sqrt{15})^2} + \frac{Y^2}{(\sqrt{6})^2} + \frac{Z^2}{(\sqrt{10})^2} = 1.$$

It is an ellipsoid with the centre at the new origin and the semi-axes are $\sqrt{15}$, $\sqrt{6}$, $\sqrt{10}$. Hence the given equation represents an ellipsoid whose centre is at $(1, -2, 1)$, the principal axes being parallel to the co-ordinate axes. The principal planes are $x = 1$, $y = -2$, $z = 1$.

Ex. 2. *Obtain the equation of the ellipsoid whose centre is at the point* $(-3, 2, -1)$ *and whose principal axes are parallel to the co-ordinate axes, the lengths of axes being* 1, 2, 3 *respectively.*

Referred to the set of rectangular axes through the point $(-3, 2, -1)$, the equation of the ellipsoid is

$$\frac{X^2}{\left(\frac{1}{2}\right)^2} + \frac{Y^2}{1^2} + \frac{Z^2}{\left(\frac{3}{2}\right)^2} = 1.$$

Hence the required equation is $4(x+3)^2 + (y-2)^2 + \frac{4}{9}(z+1)^2 = 1.$

Examples VI (C)

1. Show that the locus of a point, the sum of whose distances from the points $(a, 0, 0)$ and $(-a, 0, 0)$ is a constant is the ellipsoid of revolution.

2. Show that the equation of the ellipsoid, which has the major axis of length 12 units along the x-axis and meets the y- and z-axes at the points (0, 3, 0) and (0, 0, 2), is $x^2 + 4y^2 + 9z^2 = 36$.

3. Show that the equation of the ellipsoid referred to the axes as its principal axes and the centre as the origin and which has two of its vertices at $(\pm 1, 0, 0)$ and passes through the two points $\left(\frac{1}{2}, 1, 1\right)$ and $\left(0, \frac{1}{2}, \sqrt{\frac{7}{2}}\right)$ is $4x^2 + 2y^2 + z^2 = 4$.

4. Show that the equation $x^2 + 4y^2 - 9z^2 = 36$ represents the hyperboloid of one sheet whose centre is the origin and whose semi-axes are 6, 3, 2, the principal planes being $x = 0, y = 0, z = 0$.

5. Show that the equation $3x^2 - 4y^2 + 5z^2 + 60 = 0$ represents the hyperboloid of two sheets whose centre is the origin and whose semi-axes are $2\sqrt{5}$, $\sqrt{15}$, $2\sqrt{3}$, the principal planes being

$$x = 0, \; y = 0, \; z = 0.$$

6. Show that the surface represented by the locus of a point the difference of whose distances from the points (-4, 3, 1) and (4, 3, 1) is 6, will be a hyperboloid of two sheets.

7. (a) Show that the plane $x - 2 = 0$ intersects the ellipsoid $\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$ in an ellipse.

(b) Show that the plane $z + 1 = 0$ intersects the hyperboloid of one sheet $\frac{x^2}{32} - \frac{y^2}{18} + \frac{z^2}{2} = 1$ in a hyperbola.

8. Show that the area enclosed by the curve in which the plane $z = h$ cuts the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$\frac{\pi ab\,(c^2 - h^2)}{c^2}.$$

9. Show that the equation of the hyperboloid of one sheet if it passes through the point (1, 1, 1) and if it has the principal elliptic section $2x^2 + 4y^2 = 1$ is $2x^2 + 4y^2 - 5z^2 = 1$.

10. Show that the value of k, for which the plane $x + kz = 1$ intersects the hyperboloid of two sheets $x^2 + y^2 - z^2 + 1 = 0$ in

(i) an ellipse, is $1 < |k| < \sqrt{2}$;

(ii) a hyperbola, is $|k| < 1$.

D. Paraboloids

6.19. The elliptic paraboloid.

Let us consider the surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}. \quad \ldots \quad (1)$$

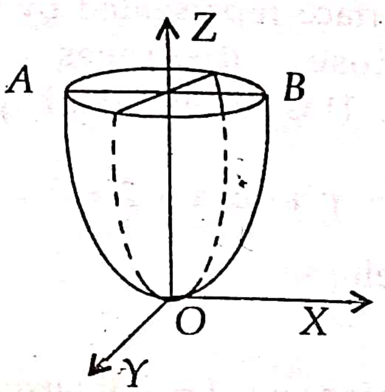

Fig. 22

(*a*) The section of (1) by a plane parallel to $z = 0$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c},\ z=k. \quad \ldots \quad (2)$$

This is an ellipse whose centre is at the point $(0, 0, k)$, if only k be positive. The ellipse increases in size with increase in k. Thus the whole surface (1) is generated by ellipses like (2) with centres on the z-axis, as k varies.

(*b*) If a point $P(x, y, z)$ lies on (1), then the point $Q(-x, y, z)$ must also lie on it. The middle point of PQ whose co-ordinates are $(0, y, z)$ lies on the yz-plane. Also the straight line PQ is perpendicular to the plane $x = 0$. Hence the plane $x = 0$ bisects all chords of (1) perpendicular to it. Thus the surface given by the equation (1) is symmetrical with respect to the yz-plane.

Similarly, it can be shown that the surface (1) is symmetrical with respect to the zx-plane. These two planes are the *principal planes* of the elliptic paraboloid (1).

(c) If k be negative, then the sections (2) are imaginary. Thus, z ...ing always positive, the surface (1) always lies on the positive side ...the plane $z = 0$. Again, since k may be any large positive quantity, ...e surface on the positive side of $z = 0$ may extend up to infinity.

(d) The section of (1) by a plane parallel to $x = 0$ is given by

$$\frac{y^2}{b^2} = \frac{2z}{c} - \frac{k^2}{a^2}, \quad x = k$$

or,

$$y^2 = \frac{2b^2}{c}\left(z - \frac{ck^2}{2a^2}\right), \quad x = k.$$

This represents a parabola whose latus rectum is $\frac{2b^2}{c}$ which is constant, showing that the sections are of same size. The vertex of the parabola is at the point $\left(k, 0, \frac{ck^2}{2a^2}\right)$ which lies on the parabola

$$\frac{x^2}{a^2} = \frac{2z}{c}, \quad y = 0$$

and is the section of (1) by the plane $y = 0$.

Thus we see that the whole surface may be formed by pushing the parabola

$$y^2 = \frac{2b^2}{c} z, \quad x = 0$$

backwards and forwards perpendicular to its plane, so that the vertex slides on the parabola

$$x^2 = \frac{2a^2}{c} z, \quad y = 0.$$

Similarly, we can show that sections of the surface by planes parallel to $y = 0$ are parabolas which can be obtained by pushing the parabola

$$x^2 = \frac{2a^2}{c} z, \quad y = 0$$

backwards and forwards, while the vertex slides on the parabola

$$y^2 = \frac{2b^2}{c} z, \quad x = 0.$$

With these considerations, the Fig. 22 represents the surface of an elliptic paraboloid.

6.20. The hyperbolic paraboloid.

Let us consider the surface represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}. \quad \ldots \quad (1)$$

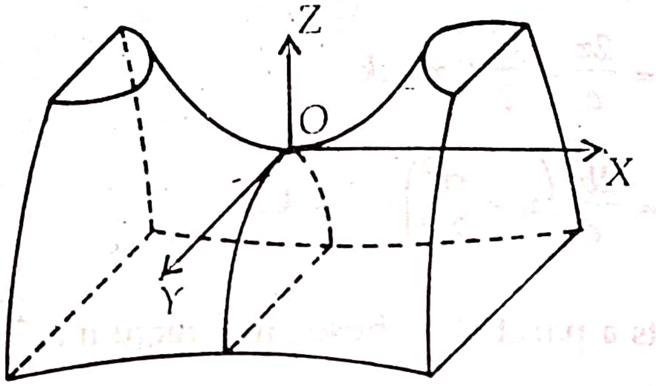

Fig. 23

(a) The section of (1) by a plane parallel to the plane $z = 0$ is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}, \; z = k. \quad \ldots \quad (2)$$

These are hyperbolas with centre lying on the axis of z. The asymptotes of the hyperbolas are parallel to the straight lines

$$\frac{x}{a} \pm \frac{y}{b} = 0, \; z = 0.$$

These are the sections of the given surface by the plane $z = 0$.

Furthermore, if k be positive, then the transverse axis of the hyperbola (2) is the x-axis ; on the other hand, if k be negative, then the transverse axis is parallel to the y-axis.

(b) As in the case of an elliptic paraboloid, the co-ordinate planes $x = 0$, $y = 0$ are the planes of symmetry and are thus the principal planes.

(c) The section of (1) by a plane parallel to $x = 0$ is given by

$$-\frac{y^2}{b^2} = \frac{2z}{c} - \frac{k^2}{a^2}, \; x = k.$$

These are parabolas.

Similarly, the section of (1) by a plane parallel to $y = 0$ is given by

$$\frac{x^2}{a^2} = \frac{2z}{c} + \frac{k^2}{b^2}, \; y = k.$$

These are also parabolas.

6.21. General equation of paraboloids.

Consider the general equation of the paraboloids

$$ax^2 + by^2 = 2cz.$$

If a and b be both positive, then this represents an elliptic paraboloid ; on the other hand, if a and b be of opposite signs, then this represents a hyperbolic paraboloid.

It is very interesting to observe that the elliptic paraboloid is the limiting form of the ellipsoid or of hyperboloid of two sheets and the hyperbolic paraboloid is the limiting form of the hyperboloid of one sheet.

The equation of the ellipsoid with the point $(-a, 0, 0)$ as origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2x}{a}.$$

Now let a, b, c all become infinite, while $\frac{b^2}{a}, \frac{c^2}{a}$ remain finite and equal to l and m respectively. Then, in the limit, equation of the ellipsoid reduces to

$$\frac{y^2}{l} + \frac{z^2}{m} = 2x,$$

which is the equation of the elliptic paraboloid.

In a similar way, the other cases can be proved.

6.22. Illustrative Examples.

Ex. 1. *Find the nature of the conicoid*

$$3x^2 - 2y^2 - 12x - 12y - 6z = 0.$$

The given equation can be written as

$$3(x-2)^2 - 2(y+3)^2 = 6(z-1)$$

or,

$$\frac{(x-2)^2}{2} - \frac{(y+3)^2}{3} = z - 1.$$

Shifting the origin to the point $(2, -3, 1)$, the equation reduces to

$$\frac{X^2}{2} - \frac{Y^2}{3} = Z.$$

It is a hyperbolic paraboloid with the principal planes $X = 0$ and $Y = 0$. Hence the given equation represents a hyperbolic paraboloid with the principal planes $x - 2 = 0$ and $y + 3 = 0$.

Ex. 2. *Find the co-ordinates of the vertex, focus and the length of the latus rectum of the principal sections of the paraboloid given by the equation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}.$$

The equation remains unchanged when (x, y, z) are changed to $(-x, y, z)$ or to $(x, -y, z)$. Therefore the surface is symmetric with respect to the yz- and xz-planes. Hence $x = 0$ and $y = 0$ are the two principal planes of the paraboloid.

The section of the surface by the plane $x = 0$ is given by

$$\frac{y^2}{b^2}=\frac{2z}{c},\ x=0\text{, that is, } y^2=\frac{2b^2}{c}z,\ x=0.$$

For this section, vertex is the origin, focus is at the point $(0, 0, \frac{b^2}{2c})$ and the length of the latus rectum is $\frac{2b^2}{c}$.

Also the section of the surface by the plane $y = 0$ is given by

$$\frac{x^2}{a^2}=\frac{2z}{c},\ y=0\text{, that is, } x^2=\frac{2a^2}{c}z,\ y=0.$$

For this section, vertex is the origin, focus is at the point $(0, 0, \frac{a^2}{2c})$ and the length of the latus rectum is $\frac{2a^2}{c}$.

Examples VI (D)

1. Show that the equation

$$2x^2+3y^2-8x+6y-12z+11=0$$

represents an elliptic paraboloid.

2. Show that the equations

$$2x=ae^{2\phi},\ y=be^{\phi}\cosh\theta,\ z=ce^{\phi}\sinh\theta$$

represent a hyperbolic paraboloid. [C. H. 1995]

3. Show that the vertex, focus and the length of the latus rectum of the principal sections of the paraboloid $4x^2-9y^2=36z$ are

$$(0, 0, 0), (0, 0, -1), 4; (0, 0, 0), \left(0, 0, \tfrac{9}{4}\right), 9.$$

4. Show that the vertex and the focus of the principal sections of the paraboloid $x^2+2y^2=4z$ are

$$(0, 0, 0), \left(0, 0, \tfrac{1}{2}\right); (0, 0, 0), (0, 0, 1).$$

5. Show that the plane $y+6=0$ intersects the hyperbolic paraboloid $\frac{x^2}{5}-\frac{y^2}{4}=6z$ in a parabola.

9 GENERATING LINES

9.1. Ruled surfaces.

We have seen earlier that cones and cylinders are surfaces, which are generated by the movement of a straight line. Surfaces, which are generated by the moving straight lines, are called *ruled surfaces*. Thus, through every point of a ruled surface, straight lines can be drawn which lie entirely on the surface. These straight lines are called *generating lines* or *generators*. In the case of a cone, the generating lines pass through a fixed point and intersect a given curve, while in the case of a cylinder, the generating lines are all parallel to a fixed line and intersect a given curve.

If, through every point on a surface, a straight line can be drawn such that it always lies wholly on the surface, then the surface is called a *ruled surface* and the straight lines are called the *generators* or *generating lines* of the surface.

We shall see presently that hyperboloids of one sheet and hyperbolic paraboloids are also generated by the movement of straight lines and hence are ruled surfaces.

A ruled surface may be (*i*) a *developable surface* in which the consecutive generating lines intersect, or (*ii*) a *skew surface* in which the consecutive generators do not intersect. The cone is an example of a developable surface and so is a cylinder. But the hyperboloid of one sheet and the hyperbolic paraboloid are examples of skew surfaces.

If, for different values of a variable real parameter, a given equation represents different members of a family of straight lines, then the equation of the ruled surface corresponding to the given system of straight lines as generators, will be obtained by eliminating the variable parameter from the equation of the family of straight lines.

9.2. An important theorem.

If a straight line meets a conicoid in three points, then the straight line lies wholly on the conicoid.

Let three points of the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)},$$

lie on the conicoid $f(x, y, z)=0$, ... (1)

that is, $f(\alpha+lr,\ \beta+mr,\ \gamma+nr)=0$. ... (2)

Since the equation (1) is of second degree in x, y, z, so also will be the equation (2) in r.

Let it be $\quad Ar^2+Br+C=0$. ... (3)

Since three points of the straight line lie on the conicoid, we have three values of r satisfying (2) and as such it should be an identity, giving $A=B=C=0$.

In this case, (3) is satisfied by all values of r, so that the straight line, which meets the conicoid in three points, wholly lies on the surface.

9.3. Condition for a straight line to be a generator of a given conicoid.

Consider the conicoid given by

$$ax^2+by^2+cz^2=1 \qquad \text{... (1)}$$

and the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)}. \qquad \text{... (2)}$$

Any point on the line (2) is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$.

If it lies on the conicoid (1), then r is given by the quadratic

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+\ (a\alpha^2+b\beta^2+c\gamma^2-1)=0. \qquad \text{... (3)}$$

If the straight line (2) be a generator of the conicoid, then it lies wholly on it, that is, (3) must be satisfied for all values of r. In other words, (3) then becomes an identity and the conditions for that are

$$al^2+bm^2+cn^2=0, \qquad \text{... (4)}$$

$$a\alpha l+b\beta m+c\gamma n=0 \qquad \text{... (5)}$$

and

$$a\alpha^2+b\beta^2+c\gamma^2=1. \qquad \text{... (6)}$$

The relation (4) shows that the lines through the centre of the conicoid, that is, the point (0, 0, 0) and parallel to the generating line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ are generators of the cone

$$ax^2+by^2+cz^2=0,$$

which is called the *asymptotic cone.*

The relation (5) shows that the generating lines, whose direction cosines are l, m, n, should lie on the plane

$$a\alpha x+b\beta y+c\gamma z=1,$$

which is the equation of the tangent plane of the conicoid at the point (α, β, γ).

The relation (6) shows that the point (α, β, γ) lies on the conicoid.

The equations (4) and (5) give two sets of values of l, m, n. This shows that through any point on the conicoid there pass two straight lines (real, coincident, or imaginary) entirely lying on the conicoid.

Cor. 1. By Lagrange's identity, we have

$$(a\alpha^2+b\beta^2)(al^2+bm^2)-(a\alpha l+b\beta m)^2=ab(\alpha m-\beta l)^2.$$

Using (4), (5) and (6), this gives

$$(1-c\gamma^2)(-cn^2)-(-c\gamma n)^2=ab(\alpha m-\beta l)^2$$

or, $$-cn^2=ab(\alpha m-\beta l)^2$$

or, $$\left(\alpha\frac{m}{n}-\beta\frac{l}{n}\right)^2=-\frac{c}{ab}.$$

From this relation, we observe that $l:m:n$ is real only if two of the coefficients a, b, c will have the positive sign while the third will have negative sign. Thus *the hyperboloid of one sheet is the only central conicoid which can be a ruled surface.*

Cor. 2. In the case of the paraboloid $ax^2+by^2=2z$, proceeding in the same way, we see that the straight line (2) will be its generator, if

$$al^2+bm^2=0,\ a\alpha l+b\beta m-n=0,\ a\alpha^2+b\beta^2-2\gamma=0.$$

From the first equation it is seen that $l:m$ will be real, only when a and b will have opposite signs. In that case, the paraboloid is a hyperbolic paraboloid. Thus *only the hyperbolic paraboloids are ruled surfaces.*

9.4. Generating lines of a hyperboloid of one sheet.

Consider the hyperboloid of one sheet given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad \ldots (1)$$

which may be put as $\left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=\left(1-\frac{y}{b}\right)\left(1+\frac{y}{b}\right)$.

Consider the straight line given by the equations

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),$$

$$\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right), \qquad \ldots (2)$$

where λ is a constant.

Eliminating λ from the equations (2), we get the hyperboloid of one sheet as in (1). Thus all the points lying on the straight line given by the equations (2) lie on the hyperboloid of one sheet (1). Consequently, the

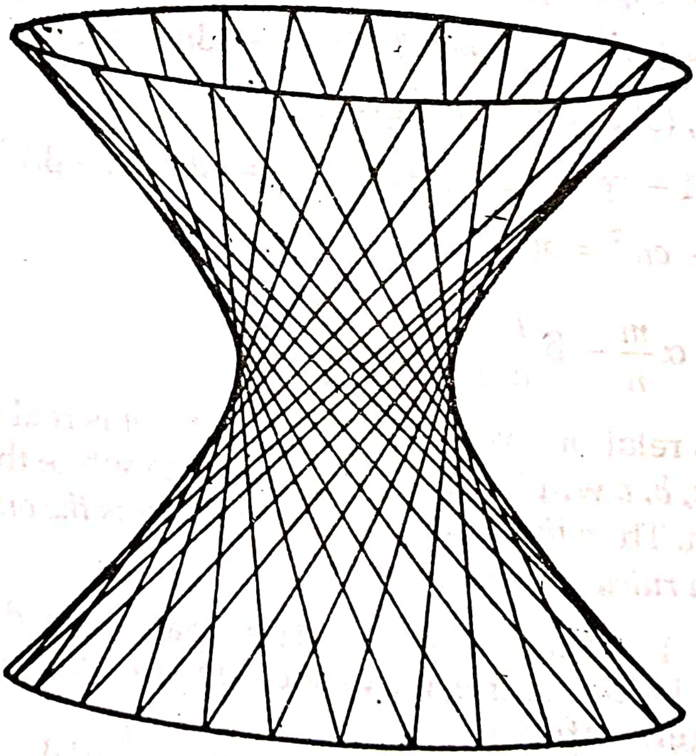

Fig. 28

straight line (2) lies wholly on the surface (1). For different values of λ, we get different straight lines all lying wholly on the surface (1), which is the hyperboloid of one sheet and as such the straight lines given by the equations (2) are the generating lines of (1).

Exactly in the similar way, we can show that the straight lines given by the equations

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\qquad \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \quad \ldots (3)$$

lie entirely on the surface (1) and hence are its generators.

Thus we see that, as λ and μ vary, we get two systems of straight lines such that every member of each system lies entirely on the hyperboloid of one sheet and hence the surface is a ruled surface having two systems of generators.

It should be observed that no values of λ and μ can be found which make (2) and (3) identical, proving that they represent entirely different systems of variable straight lines.

Note. The two systems (2) and (3) are called respectively the *λ-system* and the *μ-system* of generators.

9.5. Properties of generating lines of hyperboloid of one sheet.

(a) No two generators of the same system intersect.

Let λ_1 and λ_2 be two different values of λ, for which the two generators of the λ-system are

$$\frac{x}{a}-\frac{z}{c}=\lambda_1\left(1-\frac{y}{b}\right), \quad \ldots (1)$$

$$\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda_1}\left(1+\frac{y}{b}\right); \quad \ldots (2)$$

$$\frac{x}{a}-\frac{z}{c}=\lambda_2\left(1-\frac{y}{b}\right), \quad \ldots (3)$$

$$\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda_2}\left(1+\frac{y}{b}\right). \quad \ldots (4)$$

Subtracting (3) from (1), we get

$$(\lambda_1-\lambda_2)\left(1-\frac{y}{b}\right)=0.$$

Now, since $\lambda_1 \neq \lambda_2$, therefore $y = b$.

Again, subtracting (4) from (2), we get

$$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\left(1+\frac{y}{b}\right)=0, \text{ which gives } y=-b.$$

Thus the four equations giving two generators of the same system are not consistent and hence the two generators do not intersect.

From this, it is clear that *the hyperboloid of one sheet is a skew surface.*

(b) Any two generators of the different systems intersect.

Consider two generating lines, one of each system

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right), \quad \ldots (1)$$

$$\frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{y}{b}\right); \quad \ldots (2)$$

$$\frac{x}{a} - \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right), \quad \ldots (3)$$

$$\frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1 - \frac{y}{b}\right). \quad \ldots (4)$$

From (1) and (3), we have

$$\lambda\left(1 - \frac{y}{b}\right) = \mu\left(1 + \frac{y}{b}\right). \quad \text{Therefore } \frac{y}{b} = \frac{\lambda - \mu}{\lambda + \mu}. \quad \ldots (5)$$

Then substituting (5) in (1) and (2), we get

$$\frac{x}{a} - \frac{z}{c} = \frac{2\lambda\mu}{\lambda + \mu} \text{ and } \frac{x}{a} + \frac{z}{c} = \frac{2}{\lambda + \mu}.$$

$$\text{Therefore } \frac{x}{a} = \frac{1 + \lambda\mu}{\lambda + \mu} \text{ and } \frac{z}{c} = \frac{1 - \lambda\mu}{\lambda + \mu}. \quad \ldots (6)$$

These values of x, y, z, as given by (5) and (6), satisfy the equation (4) also, proving that the two generating lines of the two systems intersect at the point

$$\left(a\frac{1 + \lambda\mu}{\lambda + \mu}, b\frac{\lambda - \mu}{\lambda + \mu}, c\frac{1 - \lambda\mu}{\lambda + \mu}\right).$$

It can easily be verified that this point is on the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ and hence the equations (5) and (6) represent its parametric equations.

(c) Through every point of the hyperboloid of one sheet, there passes one generator of each system.

Let (α, β, γ) be a point on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \text{ so that } \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}=1. \qquad (1)$$

A generator of the λ-system will pass through the point (α, β, γ), if λ has got a value for which

$$\lambda=\frac{\frac{\alpha}{a}-\frac{\gamma}{c}}{1-\frac{\beta}{b}}=\frac{1+\frac{\beta}{b}}{\frac{\alpha}{a}+\frac{\gamma}{c}},$$

which gives $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}=1$, which is true by (1).

Similarly, a generator of the μ-system will pass through the point (α, β, γ) for which

$$\mu=\frac{\frac{\alpha}{a}-\frac{\gamma}{c}}{1+\frac{\beta}{b}}=\frac{1-\frac{\beta}{b}}{\frac{\alpha}{a}+\frac{\gamma}{c}}.$$

(d) *Two intersecting generators belonging to the two systems lie on a plane, which is tangent to the hyperboloid at their point of intersection.*

A plane through the λ-system of generators of the hyperboloid is

$$\frac{x}{a}-\frac{z}{c}-\lambda\left(1-\frac{y}{b}\right)-k\left\{\left(\frac{x}{a}+\frac{z}{c}\right)-\frac{1}{\lambda}\left(1+\frac{y}{b}\right)\right\}=0 \qquad \ldots (1)$$

and another through the μ-system of generators is

$$\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)-k'\left\{\left(\frac{x}{a}+\frac{z}{c}\right)-\frac{1}{\mu}\left(1-\frac{y}{b}\right)\right\}=0. \qquad \ldots (2)$$

It is easy to verify that when $k=k'=-\lambda\mu$, both these equations reduce to the same equation

$$\frac{x}{a}\frac{1+\lambda\mu}{\lambda+\mu}+\frac{y}{b}\frac{\lambda-\mu}{\lambda+\mu}-\frac{z}{c}\frac{1-\lambda\mu}{\lambda+\mu}=1,$$

which is also the equation of the tangent plane to the hyperboloid at the point of intersection of the λ-and μ-systems of generators.

Note. A plane through a generator is a tangent plane to the hyperboloid at some point of the generator.

9.6. Generating lines of a hyperbolic paraboloid.

Let the equation of the hyperbolic paraboloid be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z, \quad \dots (1)$$

which may be put as $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 2z.$

Consider the straight line given by the equations

$$\frac{x}{a} - \frac{y}{b} = \lambda z, \ \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda}, \quad \dots (2)$$

λ being a constant.

Eliminating λ from the two equations (2) by multiplication, we get the hyperbolic paraboloid (1). Thus all points which satisfy (2) must also satisfy (1) and hence it is proved that the straight line (2) lies entirely on (1). As λ takes different real values, the straight line (2) moves and completely generates the hyperbolic paraboloid.

Exactly in the similar manner, we can prove that the straight lines given by the equations

$$\frac{x}{a} - \frac{y}{b} = \frac{2}{\mu}, \ \frac{x}{a} + \frac{y}{b} = \mu z \quad \dots (3)$$

lie entirely on the surface (1) and as such are its generators.

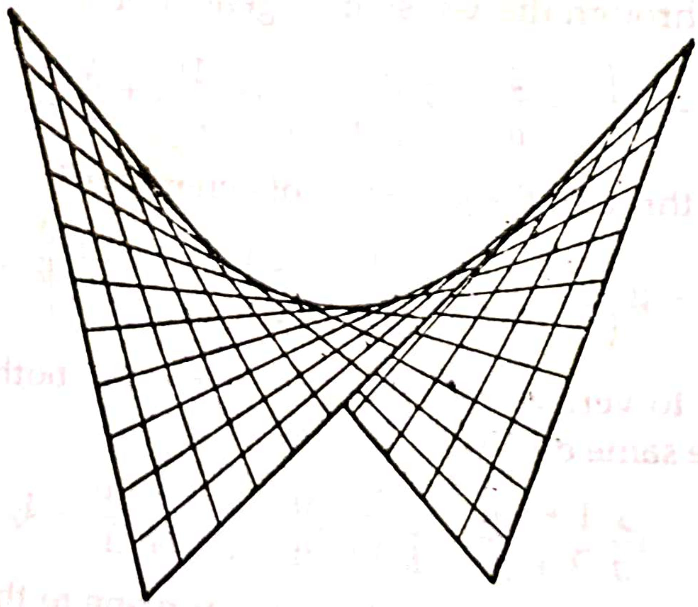

Fig. 29

Thus *the hyperboloid is a ruled surface having two systems of generating lines.*

The straight line (2), for all values of λ, is parallel to the plane $\frac{x}{a}+\frac{y}{b}=0$ and the straight line (3) is, for all values of μ, parallel to the plane $\frac{x}{a}-\frac{y}{b}=0$. Thus the generators of the hyperbolic paraboloid are parallel to one of the two fixed planes

$$\frac{x}{a}+\frac{y}{b}=0 \quad \text{or} \quad \frac{x}{a}-\frac{y}{b}=0.$$

Note. If a second degree equation represents a ruled surface having two distinct sets of generating lines, then it is either

(*i*) a hyperboloid of one sheet which is a central conicoid

or (*ii*) a hyperbolic paraboloid whose generators are parallel to a fixed plane.

9.7. Properties of generators of a hyperbolic paraboloid.

(*a*) *No two generators of the same system intersect.*

The proof is similar to that of the case of a hyperboloid of one sheet. Thus *the hyperbolic paraboloid is a skew surface.*

(*b*) *Any two generators of the different systems intersect.*

Consider two generators, one of each system,

$$\frac{x}{a}-\frac{y}{b}=\lambda z, \quad \ldots (1)$$

$$\frac{x}{a}+\frac{y}{b}=\frac{2}{\lambda}, \quad \ldots (2)$$

$$\frac{x}{a}-\frac{y}{b}=\frac{2}{\mu}, \quad \ldots (3)$$

$$\frac{x}{a}+\frac{y}{b}=\mu z. \quad \ldots (4)$$

From (1) and (3), we have

$$\lambda z=\frac{2}{\mu}, \quad \text{whence } z=\frac{2}{\lambda\mu}$$

Adding (2) and (3), we get

$$\frac{2x}{a}=\frac{2}{\lambda}+\frac{2}{\mu}, \quad \text{whence } x=a\frac{\lambda+\mu}{\lambda\mu}$$

and subtracting (3) from (2), we get

$$\frac{2y}{b}=\frac{2}{\lambda}-\frac{2}{\mu}, \quad \text{whence } y=b\frac{\mu-\lambda}{\lambda\mu}.$$

Hence the point of intersection is

$$\left(a\frac{\lambda+\mu}{\lambda\mu},\ b\frac{\mu-\lambda}{\lambda\mu},\ \frac{2}{\lambda\mu}\right).$$

For all values of λ and μ, the co-ordinates satisfy the equations of the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,$$

whose parametric equations can be put as

$$x=a\frac{\lambda+\mu}{\lambda\mu},\ y=b\frac{\mu-\lambda}{\lambda\mu},\ z=\frac{2}{\lambda\mu};$$

(c) *Through every point of the hyperbolic paraboloid, there passes one generator of each system.*

The proof is similar to that of the case of a hyperboloid of one sheet.

(d) *Two intersecting generators belonging to the two systems lie on the tangent plane to the hyperbolic paraboloid at their point of intersection.*

A plane through the λ-system of generators of the hyperbolic paraboloid is

$$\frac{x}{a}-\frac{y}{b}-\lambda z+k\left\{\left(\frac{x}{a}+\frac{y}{b}\right)-\frac{2}{\lambda}\right\}=0$$

and another plane through the μ-system is

$$\frac{x}{a}-\frac{y}{b}-\frac{2}{\mu}+k'\left\{\left(\frac{x}{a}+\frac{y}{b}\right)-\mu z\right\}=0$$

When $k=k'=\dfrac{\lambda}{\mu}$, both these planes reduce to

$$\frac{x}{a}\frac{\lambda+\mu}{\lambda\mu}-\frac{y}{b}\frac{\mu-\lambda}{\lambda\mu}=z+\frac{2}{\lambda\mu},$$

which is the equation of the tangent plane at the point of intersection of the generators.

Note. A plane through a generator is a tangent plane to the paraboloid at some point of the generator.

9.8, Illustrative Examples.

Ex. 1. *Find the equations of the generating lines of the hyperboloid* $\dfrac{x^2}{4}+\dfrac{y^2}{9}-\dfrac{z^2}{16}=1$ *which pass through the point* (2, 3, – 4).

[*N. B. H. 1979*]

Let the equations of the generator through the point (2, 3, – 4) be

$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r, \quad \ldots (1)$$

l, m, n being its direction ratios.

Any point $(2+lr, 3+mr, -4+nr)$ of it lies on the given conicoid, if

$$\tfrac{1}{4}(2+lr)^2+\tfrac{1}{9}(3+mr)^2-\tfrac{1}{16}(-4+nr)^2=1$$

or, $$r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2r\left(\frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}\right)=0.$$

The conditions that the straight line (1) is a generator, that is, it lies wholly on the conicoid are

$$\tfrac{1}{4}l^2+\tfrac{1}{9}m^2-\tfrac{1}{16}n^2=0$$

and $$\tfrac{1}{2}l+\tfrac{1}{3}m+\tfrac{1}{4}n=0.$$

Eliminating n, we get

$$\frac{l^2}{4}+\frac{m^2}{9}=\left(\frac{l}{2}+\frac{m}{3}\right)^2,$$

whence $-\frac{1}{3}lm=0$, giving either $l=0$ or $m=0$.

If $l=0$, then $\frac{m}{3}+\frac{n}{4}=0$, or, $\frac{m}{3}=\frac{n}{-4}$.

If $m=0$, then $\frac{l}{2}+\frac{n}{4}=0$, or, $\frac{l}{1}=\frac{n}{-2}$.

Hence the generators are

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \text{ and } \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}.$$

Second method :

The equations of the generating lines of the two systems are

$$\frac{x}{2}+\frac{z}{4}=\lambda\left(1+\frac{y}{3}\right), \frac{x}{2}-\frac{z}{4}=\frac{1}{\lambda}\left(1-\frac{y}{3}\right) \quad \ldots (1)$$

and $$\frac{x}{2}-\frac{z}{4}=\mu\left(1+\frac{y}{3}\right), \frac{x}{2}+\frac{z}{4}=\frac{1}{\mu}\left(1-\frac{y}{3}\right). \quad \ldots (2)$$

If the generators pass through the point (2, 3, – 4), then from (1), $\lambda=0$ from both the equations. Hence the first system of generators is given by

$$\frac{x}{2}+\frac{z}{4}=0, \; 1-\frac{y}{3}=0.$$

Similarly, from the equations (2), we get $\mu=1$.

Examples IX

1. Show that $x^2 - 9y^2 = z$ is the ruled surface generated by the family of straight lines given by

$$x + 3y - \lambda = 0, \ \lambda x - 3\lambda y - z = 0,$$

where λ is a variable parameter.

2. Find the equations of the generators of the hyperboloid

(i) $x^2 + 4y^2 - 9z^2 = 25$ passing through the point $(3, 2, 0)$.

(ii) $x^2 + y^2 - 4z^2 = 9$ passing through the point $(-3, 4, 2)$.

(iii) $9x^2 - y^2 + z^2 = 9$ passing through the point $(1, 1, 1)$.

(iv) $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 1$ passing through the point $(2, -1, \frac{4}{3})$.

(v) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ passing through the point

$(a \cos \alpha, b \sin \alpha, 0)$. [C. H. 1984 ; B. H. 1994]

3. Find the equations of the generators of the paraboloid

(i) $9x^2 - 25y^2 = 4z$ passing through the point $(1, 1, -4)$.

(ii) $4x^2 - y^2 = 7z$ passing through the point $(-2, 3, 1)$.

(iii) $4x^2 - y^2 = 8z$ passing through the point $(-3, 2, 4)$.

(iv) $16x^2 - 9y^2 = 4z$ passing through the point $(1, 0, 4)$.

4. Find the equations of the generating lines of the hyperboloid $yz + 2zx + 3xy + 6 = 0$ which pass through the point $(0, -3, 2)$.

5. Find the equations of the generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$ which pass through the point $(1, 1, 1)$. Hence find the angle between the generators. [N. B. H. 1993]

6. Prove that the generators of the surface

$$yz + zx + xy + a^2 = 0 \text{ through the point } \left(0, am, -\frac{a}{m}\right)$$

are

$$x(1 \pm m) = am - y = \mp (mz + a).$$

7. Find the equations of the generators of the hyperboloid $\dfrac{x^2}{25} + \dfrac{y^2}{16} - \dfrac{z^2}{4} = 1$ which are parallel to the plane

$$8x + 10y + 20z - 11 = 0.$$

[V. H. 1987]

8. If the straight line $\frac{x-2}{3}=\frac{y+1}{6}=\frac{z-\frac{4}{3}}{10}$ be a generator of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, then show that $a=2,\ b=3,\ c=4$.

9. (*a*) Show that the straight lines $x+1=0=z-3$ and

$$\frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$$

lie entirely on the hyperboloid $yz+2zx+3xy+6=0$.

(*b*) Show that the straight line $3x-5y=2z,\ 3x+5y=2$ lies entirely on the paraboloid $9x^2-25y^2=4z$.

(*c*) Show that the straight line $\frac{x+2}{2}=\frac{y}{3}=\frac{z-1}{-2}$ is a generator of the quadric $\frac{x^2}{4}-\frac{y^2}{9}=z$. [*C. H. 1980*]

10. Prove that any point on the line $x+1=\mu y=-(\mu+1)z$ lies on the surface $yz+zx+xy+y+z=0$ and find equations to determine the other system of lines which lie on the surface.

11. (*a*) Show that the equations $x=1+\lambda y=-1+\frac{2z}{\lambda}$ represent a generator of the hyperboloid $x^2-2yz=1$.

(*b*) Show that the equations

$$y-\lambda z+\lambda+1=0,\quad (\lambda+1)x+y+\lambda=0$$

represent, for different values of λ, generators of one system of the hyperboloid $yz+zx+xy+1=0$.

(*c*) Show that $x^2-9y^2=z$ is the ruled surface generated by the family of straight lines given by $x+3y-\lambda=0,\ \lambda x-3\lambda y-z=0$, where λ is a variable parameter. [*K. H. 2010*]

12. If the generators of $xy=bz$ include a constant angle θ, then show that their points of intersection lie on the curve of intersection of the paraboloid and the hyperboloid

$$x^2+y^2-z^2\tan^2\theta+b^2=0.$$ [*B. H. 1985*]

13. Determine the two sets of generators of the paraboloid $(2x+3y+7z)^2-(x+y+z)^2=4x-y+3z$ and prove that they are parallel respectively to the two planes

$$(2x+3y+7z)+(x+y+z)=0$$

and

$$(2x+3y+7z)-(x+y+z)=0.$$