2021

MATHEMATICS — HONOURS

First Paper

(Module – I)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Throughout the paper ℝ and ℤ denote the set of real numbers and set of integers respectively]

Group - A

(Marks : 35)

Answer ***any seven*** questions.

1. (a) If p and q are relatively primes, then find out gcd of $p+q$ and $p-q$.

(b) Correct or justify the following statement :

For any positive integer n, $f(n) = n^2 + n + 13$ is always a prime integer. 3+2

2. (a) If $a \equiv b \pmod{m}$, then show that $a^n \equiv b^n \pmod{m}$ for all positive integers n. Is the converse true? Justify your answer.

(b) Prove or disprove : $gcd(0, x) = x$ where x is a natural number. (2+1)+2

3. (a) Solve the congruence $12x \equiv 9 \pmod{15}$.

(b) State Euclid's second theorem. 3+2

4. (a) Using Chinese Remainder theorem, solve the linear congruence $9x \equiv 21 \pmod{30}$.

(b) If p is a prime integer and k is any positive integer, prove that

$$\phi\left(p^k\right) = p^k\left(1 - \frac{1}{p}\right),$$

where ϕ denotes the Euler's phi function. 3+2

5. (a) Prove that for any non-zero complex number z, $\arg z - \arg(-z) = \pm\,\pi$, according as $\arg z > 0$ or < 0, where $\arg z$ is the principal argument of z.

(b) Find all complex numbers z such that $exp(z + \bar{z}) = 1$. 3+2

6. (a) If $\cos h^{-1}(x+iy)+\cos h^{-1}(x-iy)=\cos h^{-1}a$, where x, y, a are real numbers and $a>1$, prove that the point (x, y) lies on an ellipse.

(b) Find the product of all values of $(-i)^{\frac{1}{4}}$. 3+2

7. (a) If a_1, a_2, a_3,..., a_n are n positive numbers and $a_1+a_2+a_3+\ldots+a_n=S$, then show that

$$\frac{S}{S-a_1}+\frac{S}{S-a_2}+\frac{S}{S-a_3}+\ldots+\frac{S}{S-a_n}\geq\frac{n^2}{n-1}$$

(b) If α is an imaginary root of the equation $x^7=1$, then find the value of $(\alpha^6+1)(\alpha^5+1)(\alpha^4+1)(\alpha^3+1)(\alpha^2+1)(\alpha+1)$. 3+2

8. Using Sturm's functions, show that the roots of $x^4+4x^3-x^2-10x+3=0$ are all real and distinct. 5

9. Solve $x^4+3x^3+5x^2+4x+2=0$ by Ferrari's method. 5

10. Calculate Sturm's functions and find the number and nature of real roots of the equation $x^5-5x+2=0$. 3+1+1

11. Show that all the imaginary roots of the equation $x^7=1$ are special roots. If α is a special root of $x^7=1$, form the equation whose roots are $\alpha+\alpha^6$, $\alpha^2+\alpha^5$ and $\alpha^3+\alpha^4$. 2+3

12. (a) If α, β, γ are the roots of the equation $ax^3+3bx^2+3cx+d=0$, find the value of

$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

(b) Find the value of k, for which the equation $x^4+4x^3-2x^2-12x+k=0$ has 4 real and unequal roots. 3+2

13. Show that if the roots of the equation $x^4+x^3-4x^2-3x+3=0$ are increased by 2, the transformed equation is a reciprocal equation. Solve the reciprocal equation and hence obtain the solution of the given equation. 2+3

Group - B

(Marks : 15)

Answer ***any three*** questions.

14. (a) Prove of disprove : If X, Y and Z are subsets of a set S, then $X\Delta Z=X\Delta Y$ implies $Z=Y$.

(b) Let $A=\{1, 2, 3, 4\}$ and a relation ρ on A is given by $\rho=\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (3, 2), (2, 1), (4, 1), (3, 1)\}$. Verify whether the relation is an equivalence relation. 3+2

15. (a) If $f:A\to B$ and $g:B\to C$ are two mappings such that $g\circ f:A\to C$ is surjective.

Verify whether g is surjective. Is it necessary that f is surjective? Justify your answer.

(b) Find two mappings f and g such that $f\circ g\neq g\circ f$. (2+1)+2

16. (a) Prove that a finite semigroup in which both cancellation laws hold is a group.

(b) Does the set $M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$ form a group with respect to matrix multiplication? Justify your answer. 3+2

17. (a) If each element of a group G is its own inverse, then prove that G is abelian. Is the converse true? Justify your answer.

(b) Give an example of a group $(G, \circ)$ in which $\mathrm{o}(a) \cdot \mathrm{o}(b) \neq \mathrm{o}\,(a \circ b)$, for some $a, b \in G$; where $\mathrm{o}(a)$ means order of the element a in G. (2+1)+2

18. (a) In a group $(G, \circ)$, $\mathrm{o}(a) = 5$ and $a \circ b \circ a^{-1} = b^2$. Show that if $b \neq e$ (the identity element of G), then $\mathrm{o}(b) = 31$.

(b) If $(H, \cdot)$ is a subgroup of $(G, \cdot)$, show that $H^{-1} = H$, where $H^{-1} = \{a^{-1} : a \in H\}$. 3+2
