2021

MATHEMATICS — HONOURS

Paper : CC-5

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real numbers.

Group – A

(Marks : 20)

1. Answer the following multiple choice questions having only one correct option. Choose the correct option and justify : (1+1)×10

(a) $\lim\limits_{x\to 0} \dfrac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} =$

(i) 0 (ii) 1 (iii) $\dfrac{1}{2}$ (iv) does not exist.

(b) $\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{1}{x}}{x} + x\sin\dfrac{1}{x}\right) =$

(i) 2 (ii) 0 (iii) does not exist (iv) 1.

(c) f is defined in (0, 4) by $f(x) = 2x - 2[x]$. Then

(i) f is continuous at $x = 1$ (ii) f is monotone decreasing in (0, 4)

(iii) f is not continuous at $x = 1$ (iv) f is constant in (0, 4).

(d) Which of the following functions has finite number of points of discontinuity in $\mathbb{R}$?

(i) $\tan x$ (ii) $x[x]$

(iii) $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ (iv) $\sin[\pi x]$.

(e) A real-valued continuous function f assumes only irrational values in [1, 2] and $f(1\cdot 5)=\sqrt{\pi}$, then

(i) $f(x) = ½$ everywhere in [1, 2]

(ii) $f(x) = 0$, everywhere in [1, 2]

(iii) $f(x)=\sqrt{\pi}$, everywhere in [1, 2]

(iv) $f(x) = \pi$, everywhere in [1, 2].

(f) $f(x) = x^2$, $x \in \mathbb{R}$, then

(i) f is uniformly continuous in (a, ∞), $a \in \mathbb{R}$

(ii) f is not continuous in (a, ∞), $a \in \mathbb{R}$

(iii) f is constant in (a, ∞), $a \in \mathbb{R}$

(iv) f is uniformly continuous in $[a, b]$ but not uniformly continuous in (a, ∞), where $-\infty < a, b < \infty$.

(g) A function f is defined in [–1, 1] by $f(x)=\begin{cases} 1-x^2 & \text{for } -1\le x<0 \\ x^2+x+1 & \text{for } 0\le x\le 1 \end{cases}$.

Then

(i) $f'(x) = 0$ at $x = 0$ (ii) $f'(x) = 1$ at $x = 0$

(iii) f is not differentiable at $x = 0$ (iv) f is not continuous at $x = 0$.

(h) $f(x) = x^x$, $x > 0$, then

(i) $f(x)$ has a local maximum at $x = 1/e$

(ii) $f(x)$ has a local minimum at $x = e$

(iii) $f(x)$ has neither a local minimum nor a local maximum at $x = 1/e$

(iv) $f(x)$ has local minimum at $x = 1/e$.

(i) $\lim\limits_{x\to 0} \dfrac{x-\tan x}{x^3} =$

(i) $-1/2$ (ii) $1/3$ (iii) $+\frac{1}{2}$ (iv) $-1/3$.

(j) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that $f'(x) \neq 0$ $\forall x\in(a, b)$. Then on $[a, b]$

(i) f is either increasing or decreasing.

(ii) f is neither increasing nor decreasing.

(iii) f is a constant function

(iv) $f(x) = 0$ has no root.

Group – B

(Marks : 25)

Answer ***any five*** questions.

2. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = A > 0$ and $\lim_{x\to c} g(x) = \infty$ for some $c \in \mathbb{R}$.

Prove that $\lim_{x\to c} [f(x)\, g(x)] = \infty$.

(b) Use the definition of limit to show that $\lim_{x\to\infty} \frac{x-[x]}{x} = 0$. 3+2

3. (a) Apply Sandwich theorem to evaluate $\lim_{x\to 0} (1+x)^{1/x}$.

(b) If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(x) > 0 \ \forall x \in [a,b]$. Prove that there exists $\alpha > 0$ such that $f(x) \geq \alpha \ \forall x \in [a,b]$. 3+2

4. (a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0$, $f(b) > 0$ and $A = \{x \in [a,b] : f(x) < 0\}$. If $w = \sup A$, prove that $f(w) = 0$.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at '0' and f has discontinuity of second kind at every other point in $\mathbb{R}$. 3+2

5. Discuss the continuity of $f(x)$ for $x \geq 0$, where

$$f(x) = \begin{cases} 0, & \text{when } x = 0 \\ \frac{1}{x}, & \text{when } 0 < x < 1 \\ \frac{1}{x^2}\sin\frac{\pi x}{2}, & 1 \leq x \leq 2 \\ 1 - e^{2-x}, & \text{for } x > 2 \end{cases}$$

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6. (a) Evaluate : $\lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right)^x$

(b) Prove that if $f(x)$ is continuous at $x = a$ and for every $\delta > 0$ there is a point $c_\delta \in (a - \delta, a + \delta)$ such that $f(c_\delta) = 0$, then $f(a) = 0$. 2+3

7. (a) Show that $f(x)=\frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ for $a > 0$ but not uniformly continuous on $(0, \infty)$.

(b) Examine uniform continuity of $\cos\frac{1}{x}$ on (0, 2). 3+2

8. Prove or disprove : Monotonic decreasing function on $\mathbb{R}$ cannot have jump discontinuity. 5

9. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A=\{x \in \mathbb{R} : f(x)>0\}$. If $c \in A$, show that there exists a neighbourhood N_c of c such that $N_c \subseteq A$.

Using this result, show that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x)=(-1)^{[2x]}$ is discontinuous at '1'. $[x]$ denotes the largest integer not exceeding x. 3+2

Group – C

(Marks : 20)

Answer ***any four*** questions.

10. (a) Let f be a real valued function on a closed and bounded interval $[a, b]$. If $f'(c) > 0$ for some $c \in (a, b)$, prove that f is increasing at $x = c$.

(b) Let $f(x)=x^5+4x+1$, $x \in \mathbb{R}$. Show that f has a inverse function g which is differentiable on $\mathbb{R}$. Also find $g'(1)$. 3+2

11. (a) Let $\phi(x)=f(x)+f(1-x)$ and $f''(x)<0$ for all $x \in [0, 1]$. Prove that ϕ is increasing in $0 \le x \le \frac{1}{2}$ and decreasing in $\frac{1}{2} \le x \le 1$.

(b) Show that if two functions have equal derivative at every point of (a, b), then they differ only by constant. 3+2

12. (a) Prove that $\log(1 + x)$ lies between $x-\frac{x^2}{2}$ and $x-\frac{x^2}{2(1+x)}$, for all $x > 0$.

(b) Show that $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta$, where $0<\alpha<\theta<\beta<\pi/2$. 3+2

13. (a) Let f be a real valued function on the interval I such that f' exists and bounded on I. Prove that f is uniformly continuous on I.

(b) Give an example of a uniform continuous function on [0, 1] which is differentiable on (0, 1) but the derived function is unbounded on (0, 1). 3+2

14. State and prove Darboux's theorem on derivatives. 5

15. (a) Where do the function $\sin 3x - 3\sin x$ attain local maximum or local minimum values in $(0, 2\pi)$?

(b) Evaluate $\lim\limits_{x \to 1-} \dfrac{\log(1-x)}{\cot(\pi x)}$. 3+2

16. If the sum of the lengths of the hypotenuse and the another side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between these sides is $\pi/3$. 5
